

# $A_\infty/L_\infty$ structure and alternative action for WZW-like superstring field theory

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## Abstract

We propose new gauge invariant actions for open NS, heterotic NS, and closed NS-NS superstring field theories. They are based on the large Hilbert space, and have Wess-Zumino-Witten-like expressions which are the  $\mathbb{Z}_2$ -reversed versions of the conventional WZW-like actions. On the basis of the procedure proposed in arXiv:1505.01659, we show that our new WZW-like actions are completely equivalent to  $A_\infty/L_\infty$  actions proposed in arXiv:1403.0940 respectively.

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## 1 Introduction and Summary

Recently, the formulation of superstring field theory has progressed gradually: Actions for superstring field theories were constructed and their properties have been clarified [1–4]. In addition to these, 1PI effective field theory approach has provided a good insight into self-dual gauge theory and supergravity [5]. One trigger of these developments was given by the establishment of the  $A_\infty/L_\infty$  formulation: the gauge invariant actions for open NS string [6], heterotic NS string, and closed NS-NS string [7] were constructed by giving the prescription to get the string products satisfying  $A_\infty/L_\infty$ -relations. This prescription was extended to the cases including the Ramond sector and the equations of motion were provided in the work of [8]. At least at the tree level, these theories reproduce S-matrices of perturbative superstring theory with insertions of picture-changing operators at external lines [9]. One of our aims in this paper is to develop the understanding of the relation between these  $A_\infty/L_\infty$  theories and other theories such as WZW-like theories<sup>1</sup>.

While these  $A_\infty/L_\infty$  theories work well, we know that WZW-like theories [16–21] would also give the same results from [22,23]<sup>2</sup>. Actually, we have clear understandings for open superstrings:

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<sup>1</sup>For other approaches, see also [10–15].

<sup>2</sup>For the R sector, see [24–28].

for the NS sector, the relation between the  $A_\infty$  action given in [6] and the Berkovits WZW-like action [16] is clarified by the works of [29–31], and for the NS and R sectors, the equivalence of the complete action from which the equations of motion given in [8] are derived and the complete action proposed in [2] is provided by the work of [4]. It would be important to extend these understandings of open strings to the case of closed strings and to construct complete actions for heterotic and type II theories. However, it seems to be difficult to discuss them on the basis of the same procedure<sup>3</sup> as [29] and known WZW-like actions [18–21]: It necessitates other insights because of the existence of *the skew* between  $A_\infty/L_\infty$  actions and these conventional WZW-like actions, which we explain. It would be helpful to consider their dual versions.

Recall that the Berkovits theory is formulated on the large Hilbert space, which is the state space whose superconformal ghost sector is spanned by  $\xi(z)$ ,  $\eta(z)$ , and  $\phi(z)$  [33]. In this paper, we write  $\eta$  for the zero mode of  $\eta(z)$  for brevity. An NS string field  $\Phi$  of the Berkovits theory is a Grassmann even, ghost number 0, and picture number 0 state in the large Hilbert space:  $\eta \Phi \neq 0$ , and all string products are defined by Witten’s star product [34]. Using a real parameter  $t \in [0, 1]$  and a path  $\Phi(t)$  connecting  $\Phi(0) = 0$  and  $\Phi(1) = \Phi$ , the action of the Berkovits theory is given by *the Wess-Zumino-Witten-type form*

$$S_{\text{Berkovits}} = \langle e^{-\Phi} \eta e^{\Phi}, e^{-\Phi} Q e^{\Phi} \rangle + \int_0^1 dt \langle e^{-\Phi(t)} \partial_t e^{\Phi(t)}, \llbracket e^{-\Phi(t)} \eta e^{\Phi(t)}, e^{-\Phi(t)} Q e^{\Phi(t)} \rrbracket^* \rangle,$$

where  $Q$  is the BRST operator,  $\langle A, B \rangle$  is the BPZ inner product of  $A$  and  $B$ , and  $\llbracket A, B \rrbracket^*$  is the graded commutator for the star products of  $A$  and  $B$ . One can find that this action has nonlinear gauge invariances given by

$$\delta e^{\Phi} = e^{\Phi} (\eta \Omega_B) + (Q \Lambda_B) e^{\Phi},$$

where  $\Lambda_B$  and  $\Omega_B$  is gauge parameters.

#### Conventional WZW-like form

It was shown in [19] that Berkovits action can be written as *a WZW-like form*

$$S_{\text{Berkovits}} = \int_0^1 \langle \tilde{A}_t[\Phi(t)], \eta \tilde{A}_Q[\Phi(t)] \rangle$$

with *WZW-like functionals* of the dynamical string field  $\Phi$  defined by

$$\tilde{A}_Q[\Phi(t)] \equiv e^{-\Phi(t)} (Q e^{\Phi(t)}), \quad \tilde{A}_t[\Phi(t)] \equiv e^{-\Phi(t)} (\partial_t e^{\Phi(t)}).$$

The equation of motion is given by the  $t$ -independent form

$$\eta \tilde{A}_Q[\Phi] = 0,$$

and we can represent gauge transformations using nilpotent operators  $Q_{\tilde{A}_Q[\Phi]}$  and  $\eta$ :

$$\tilde{A}_\delta[\Phi] = Q_{\tilde{A}_Q[\Phi]} \tilde{\Lambda} + \eta \tilde{\Omega},$$

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<sup>3</sup>By taking another approach, namely by constructing an  $L_\infty$ -morphism connecting their equations of motion, one can discuss the *on-shell equivalence* of  $L_\infty$  actions [7] and conventional WZW-like actions [19, 21]. The existence of the  $L_\infty$ -morphism provides the equivalence of the solution spaces of two equations of motion up to  $Q$ -exact terms, which implies the on-shell states match each other. See [32] for more details.

where  $\tilde{\Lambda} \equiv e^\Phi \Lambda_B e^{-\Phi}$  and  $\tilde{\Omega} \equiv \Omega_B$  are redefined gauge parameters and

$$\tilde{A}_\delta[\Phi] \equiv e^{-\Phi}(\delta e^\Phi), \quad Q_{\tilde{A}_Q[\Phi]} \tilde{\Lambda} \equiv Q \tilde{\Lambda} + m_2(\tilde{A}_Q, \tilde{\Lambda}) + m_2(\tilde{\Lambda}, \tilde{A}_Q).$$

We write  $m_2(A, B)$  for the star product of  $A$  and  $B$ . A significant feature of this WZW-like form of the action is that one can obtain all properties of the action by using not explicit forms of functionals  $\tilde{A}_Q[\Phi]$ ,  $\tilde{A}_t[\Phi]$  but only specific algebraical relations of these functionals, which we call *WZW-like relations*:

$$Q \tilde{A}_Q[\Phi] + m_2(\tilde{A}_Q[\Phi], \tilde{A}_Q[\Phi]) = 0,$$

$$(-)^d d \tilde{A}_Q[\Phi] = Q_{\tilde{A}_Q[\Phi]} \tilde{A}_d[\Phi],$$

where  $d = \partial_t, \delta, \eta$  and the upper index of  $(-)^A$  denotes the Grassmann parity of  $A$ . While the first relation provides *the constraint* for  $\tilde{A}_Q[\Phi]$ , the second relation specifies its derivatives and the properties of the equation of motion. We call  $\tilde{A}_d[\Phi]$  *an associated field*.

The first relation implies that the WZW-like functional  $\tilde{A}_Q[\Phi]$ , which we call *a pure-gauge-like field*<sup>4</sup>, gives a solution of the Maurer-Cartan equation for the bosonic open string products

$$\mathbf{M}^B = Q + m_2,$$

which consists of the BRST operator  $Q$  and the star product  $m_2$ . Recall that  $\mathbf{M}^B$  satisfies  $A_\infty$ -relations:  $Q^2 = 0$ ,  $Q m_2(A, B) - m_2(QA, B) - (-)^A m_2(A, QB) = 0$ , and  $m_2(m_2(A, B), C) - m_2(A, m_2(B, C)) = 0$ . The derivation properties for  $d = \partial_t, \delta, \eta$  also hold:  $[[d, Q]] = 0$  and  $d m_2(A, B) - m_2(dA, B) - (-)^{dA} m_2(A, dB) = 0$ . In this paper, we symbolically write

$$[[\mathbf{M}^B, \mathbf{M}^B]] = 0, \quad [[d, \mathbf{M}^B]] = 0$$

for the  $A_\infty$ -relations of  $\mathbf{M}^B$  and the  $d$ -derivation properties of  $\mathbf{M}^B$ . These properties yield the WZW-like relations, and then, the constraint equation and the equation of motion are given by

$$\text{Constraint (C-WZW)} : \quad Q \tilde{A}_Q[\Phi] + m_2(\tilde{A}_Q[\Phi], \tilde{A}_Q[\Phi]) = 0,$$

$$\text{E.O.M. (C-WZW)} : \quad \eta \tilde{A}_Q[\Phi] = 0.$$

This construction is extended to more generic case: open NS superstrings with stubs, heterotic NS strings [19], and closed NS-NS strings [21]. As an example, we consider the Berkovits theory with stubs [35]. The starting point is a set of generic bosonic open string products

$$\mathbf{M}_{\text{stub}}^B = Q + m_2^{\text{st}} + m_3^{\text{st}} + m_4^{\text{st}} + \dots$$

which are nonassociative but satisfy  $A_\infty$ -relations and derivation properties for  $d = \partial_t, \delta, \eta$ :

$$[[\mathbf{M}_{\text{stub}}^B, \mathbf{M}_{\text{stub}}^B]] = 0, \quad [[d, \mathbf{M}_{\text{stub}}^B]] = 0.$$

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<sup>4</sup>The Maurer-Cartan equation for  $\mathbf{M}^B$  and the explicit form of  $\tilde{A}_Q[\Phi]$  have the same forms as the equation of motion and pure gauge field in bosonic open string field theory respectively. Actually, one can construct  $\tilde{A}_Q[\Phi] \equiv \tilde{A}_Q[1; \Phi]$  as the  $\tau = 1$  value solution  $\tilde{A}_Q[\tau; \Phi]$  of the differential equation  $\partial_\tau \tilde{A}_Q[\tau; \Phi] = Q_{\tilde{A}_Q[\tau; \Phi]} \tilde{A}_Q[\tau; \Phi]$  with the initial condition  $\tilde{A}_Q[0; \Phi] = 0$ , which is just the defining equation of the pure gauge in bosonic theory.

Using these  $M_{\text{stub}}^{\text{B}}$ ,  $d$ , and the NS open string field  $\varphi$ , one can construct a pure-gauge-like field  $\tilde{A}_Q^{\text{st}}[\varphi]$  and an associated field  $\tilde{A}_d^{\text{st}}[\varphi]$ . The resultant theory is given by the following action

$$S_{\text{stub}} = \int_0^1 dt \langle \tilde{A}_t^{\text{st}}[\varphi], \eta \tilde{A}_Q^{\text{st}}[\varphi] \rangle,$$

and characterized by the pair of equations:

$$\text{Constraint (C-WZW)} : \quad Q \tilde{A}_Q^{\text{st}}[\varphi] + m_2^{\text{st}}(\tilde{A}_Q^{\text{st}}[\varphi], \tilde{A}_Q^{\text{st}}[\varphi]) + \sum_{n=3}^{\infty} m_n^{\text{st}}(\overbrace{\tilde{A}_Q^{\text{st}}[\varphi], \dots, \tilde{A}_Q^{\text{st}}[\varphi]}^n) = 0,$$

$$\text{E.O.M. (C-WZW)} : \quad \eta \tilde{A}_Q^{\text{st}}[\varphi] = 0.$$

However, when we compare this conventional WZW-like action and the  $A_{\infty}$  action given in [7], we notice that there exists a skew:  $A_{\infty}$  theory is characterized by the pair of equations

$$\begin{aligned} \text{Constraint } (A_{\infty}) : \quad & \eta \Psi = 0, \\ \text{E.O.M. } (A_{\infty}) : \quad & Q \Psi + M_2(\Psi, \Psi) + \sum_{n=3}^{\infty} M_n(\overbrace{\Psi, \dots, \Psi}^n) = 0, \end{aligned} \tag{1.1}$$

where  $\Psi$  is the NS open string field of  $A_{\infty}$  theory and  $\{M_n\}_{n=2}^{\infty}$  are NS superstring products. While the BRST operator  $Q$  and the string products  $\{m_n^{\text{st}}\}_{n=2}^{\infty}$  determine the constraint equation for the ingredients of the conventional WZW-like theory, that of  $A_{\infty}$  theory is supplied by  $\eta$ . This skew is just the obstacle to obtain the off-shell equivalence of two theory by the naive way.

### Alternative WZW-like form

It is known that Berkovits action can be also written as *alternative WZW-like form*

$$S_{\text{Berkovits}} = - \int_0^1 dt \langle A_t[\Phi(t)], Q A_{\eta}[\Phi(t)] \rangle \tag{1.2}$$

with *alternative WZW-like functionals* of the dynamical string field  $\Phi$  defined by

$$A_{\eta}[\Phi] \equiv (\eta e^{\Phi(t)})e^{-\Phi(t)}, \quad A_t[\Phi] \equiv (\partial_t e^{\Phi(t)})e^{-\Phi(t)}. \tag{1.3}$$

Then, the equation of motion is given by the  $t$ -independent form

$$Q A_{\eta}[\Phi] = 0,$$

and we can represent gauge transformations using nilpotent operators  $D_{\eta}^*$  and  $Q$ :

$$A_{\delta}[\Phi] = D_{\eta}^* \Omega + Q \Lambda,$$

where  $\Lambda \equiv \Lambda_{\text{B}}$  and  $\Omega \equiv e^{-\Phi} \Omega_{\text{B}} e^{\Phi}$  are redefined gauge parameters and

$$A_{\delta}[\Phi] \equiv (\delta e^{\Phi})e^{-\Phi}, \quad D_{\eta}^* \Omega \equiv \eta \Omega - m_2(A_{\eta}[\Phi], \Omega) - (-)^{\Omega} m_2(\Omega, A_{\eta}[\Phi]).$$

A significant feature of this alternative WZW-like form is that, as the conventional case, one can obtain all properties of the action by using only specific algebraical relations of alternative WZW-like functionals, which we also call *WZW-like relations*:

$$\begin{aligned}\eta A_\eta[\Phi] - m_2(A_\eta[\Phi], A_\eta[\Phi]) &= 0, \\ (-)^d d A_\eta[\Phi] &= D_\eta^* A_d[\Phi],\end{aligned}\tag{1.4}$$

where  $d = \partial_t, \delta, Q$ . Therefore, we can say that this type of WZW-like theory also belongs to the category of so-called the WZW-like formulation. While the first relation provides the constraint for  $A_\eta[\Psi]$ , the second relation specifies the properties of the equation of motion.

Note that this type of WZW-like theory, a dual version of the conventional WZW-like theory, is characterized by the pair of equations

$$\begin{aligned}\text{Constraint (A-WZW)} : \quad & \eta A_\eta[\Phi] - m_2(A_\eta[\Phi], A_\eta[\Phi]) = 0, \\ \text{E.O.M. (A-WZW)} : \quad & Q A_\eta[\Phi] = 0.\end{aligned}$$

One can find that there is no skew between this type of WZW-like theory and  $A_\infty$  theory. Actually, as demonstrated in [29], by decomposing the NS superstring product  $\mathbf{M} = \mathbf{Q} + \mathbf{M}_2 + \mathbf{M}_3 + \dots$  given in [6] as  $\mathbf{M} = \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}$  and using a redefined string field  $A_\eta[\Psi] \equiv \pi_1 \widehat{\mathbf{G}}(1 - \Psi)^{-1}$ , we can transform the pair of equations (1.1) characterizing the  $A_\infty$  action into

$$\begin{aligned}\text{Constraint } (A_\infty) : \quad & \eta A_\eta[\Psi] - m_2(A_\eta[\Psi], A_\eta[\Psi]) = 0, \\ \text{E.O.M. } (A_\infty) : \quad & Q A_\eta[\Psi] = 0.\end{aligned}$$

Therefore, we expect that one can construct WZW-like actions which are off-shell equivalent to  $A_\infty$  and  $L_\infty$  actions proposed in [7] as this type of WZW-like theories.

### Main results of this paper

What is the starting point of this type of WZW-like theory? We can read it from the constraint for  $A_\eta[\Phi]$  or  $A_\eta[\Psi]$ . For the Berkovits theory, it is given by the following  $A_\infty$ -product:

$$D_*^\eta \equiv \eta - m_2,$$

which we call *the dual products* of  $\mathbf{M}$  in [6]. We write  $D_2^\eta$  for  $-m_2$ . Then, one can check that  $\eta + D_2^\eta$  satisfies the  $A_\infty$ -relations of  $\eta$ :  $\eta^2 = 0$ ,  $\eta D_2^\eta(A, B) - D_2^\eta(\eta A, B) - (-)^A D_2^\eta(A, \eta B) = 0$ , and  $D_2^\eta(D_2^\eta(A, B), C) - D_2^\eta(A, D_2^\eta(B, C)) = 0$ . The derivation properties for  $d = \partial_t, \delta, Q$  also hold:  $\llbracket d, \eta \rrbracket = 0$  and  $d D_2^\eta(A, B) - D_2^\eta(dA, B) - (-)^A D_2^\eta(A, dB) = 0$ . We write them as follows:

$$\llbracket D_*^\eta, D_*^\eta \rrbracket = 0, \quad \llbracket d, D_*^\eta \rrbracket = 0.$$

Actually, one can construct the pure-gauge-like field  $A_\eta[\Phi] = (\eta e^\Phi) e^{-\Phi}$  of (1.3) and the action (1.2) using this  $D_*^\eta = \eta - m_2$ : the pure-gauge-like field  $A_\eta[\Phi]$  in (1.3) is given by the  $\tau = 1$  value  $A_\eta[\Phi] \equiv A_\eta[\tau = 1; \Phi]$  of the solution  $A_\eta[\tau; \Phi]$  of the differential equation

$$\frac{\partial}{\partial \tau} A_\eta[\tau; \Phi] = \eta \Phi - m_2(A_\eta[\tau; \Phi], \Phi) + m_2(\Phi, A_\eta[\tau; \Phi])$$

with the initial condition  $A_\eta[\tau = 0; \Phi] = 0$ , where  $\tau \in [0, 1]$  is a real parameter.

In this paper, we show that a nonassociative extended version of this construction gives our new WZW-like theory. The starting point is the following  $A_\infty$  products

$$D^\eta = \eta + D_2^\eta + D_3^\eta + D_4^\eta + \dots$$

satisfying the  $A_\infty$ -relations of  $\eta$  and derivation properties for  $d = \partial_t, \delta, Q$ :

$$[[D^\eta, D^\eta]] = 0, \quad [[d, D^\eta]] = 0.$$

One can construct these products by the dual products of  $M = \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}$  given in [7], namely  $D^\eta \equiv \widehat{\mathbf{G}} \eta \widehat{\mathbf{G}}^{-1}$ . Then, using these dual products  $D^\eta$ , we propose alternative WZW-like actions which are equivalent to  $A_\infty$  actions proposed in [7] as this type of WZW-like theories

$$S_\eta = - \int_0^1 dt \langle A_t[\varphi(t)], Q A_\eta[\varphi(t)] \rangle, \quad (1.5)$$

where  $\varphi$  is a dynamical NS string field,  $t \in [0, 1]$  is a real parameter. The WZW-like functionals  $A_\eta[\varphi]$  and  $A_t[\varphi]$  satisfy the nonassociative extended versions of (1.4):

$$\eta A_\eta[\varphi] - m_2(A_\eta[\varphi], A_\eta[\varphi]) + \sum_{n=3}^{\infty} D_n^\eta \left( \overbrace{A_\eta[\varphi], \dots, A_\eta[\varphi]}^n \right) = 0, \quad (1.6)$$

$$(-)^d d A_\eta[\varphi] = D_\eta A_d[\varphi],$$

where now the  $A_\eta$ -shifted dual product  $D_\eta$  is given by

$$D_\eta \Omega \equiv \eta \Omega - m_2(A_\eta[\varphi], \Omega) - (-)^\Omega m_2(\Omega, A_\eta[\varphi]) + \sum_{\text{cyclic } n=2}^{\infty} \sum_{n=2}^{\infty} (-)^{\text{sign}} D_{n+1}^\eta \left( \overbrace{A_\eta[\varphi], \dots, A_\eta[\varphi]}^n, \Omega \right).$$

Note that this WZW-like action for generic open NS strings just reduces to the alternative WZW-like form (1.2) of the Berkovits theory when we take the star product. We would like to emphasize that we do not need a specific form of  $A_\eta[\varphi]$  or  $A_t[\varphi]$  as a functional of given dynamical string field  $\varphi$  but only their properties (1.6) to show the properties of the action: its variation, equations of motion, gauge invariance, and so on. In appendix D, we will give explicit forms of two realizations of these functionals  $A_\eta$  and  $A_d$  using two different dynamical string fields:  $\Psi$  in the small Hilbert space and  $\Phi$  in the large Hilbert space.

In the above, we take open NS theory as an example to grab a feature of our alternative WZW-like approach and its necessity. The details of the above open NS theory are discussed in appendix D. In the following sections, our main topic is “heterotic NS theory”: We consider the  $L_\infty$  action and explain how its WZW-like properties arise. On the basis of the WZW-like structure naturally arising from  $L_\infty$  actions, we propose new gauge invariant actions for heterotic NS and NS-NS strings, as well as that for open NS strings with stubs which we introduced in (1.5), which are  $\mathbb{Z}_2$ -reversed versions of the conventional ones [19, 21]. It is shown that they are completely equivalent to the  $A_\infty/L_\infty$  actions proposed in [7].

We expect that our new WZW-like actions would provide a first step to construct complete actions for heterotic and type II string field theories. Actually, an action for open superstring field theory including the R sector was constructed [2] by starting with this type of WZW-like action: The R string field couples to the Berkovits theory for the NS sector gauge-invariantly on the basis of (not the conventional but) this type of WZW-like gauge structure. We would like to mention that although we expect that our new WZW-like actions are also equivalent to the conventional WZW-like actions, the all order equivalence of these has not been proven: we will show lower-order equivalence to the conventional WZW-like actions only.

This paper is organized as follows. In section 2, after a brief review of the  $L_\infty$  formulation, we clarify a WZW-like structure naturally arising from it. We show that the  $L_\infty$  action can be written in the WZW-like form: the functionals which appear in the action satisfy the  $\mathbb{Z}_2$ -reversed versions of the conventional WZW-like relations given in [19], which guarantees the gauge invariance of the action. The on-shell condition and the gauge transformation of the WZW-like action are derived by using only the WZW-like relations. We also see how the gauge parameters appearing in the WZW-like form are parameterized by that in the  $L_\infty$ -type form. Then, we conclude the  $L_\infty$  action gives one realization of the WZW-like action  $S_\eta[\Phi]$  parametrized by its dynamical string field  $\Phi$ . In section 3, we provide the alternative parameterization of the WZW-like action using the string field  $V$  in the large Hilbert space. The functionals satisfying the WZW-like relations can be defined by the differential equations which are the  $\mathbb{Z}_2$ -reversed versions of those given in [19]. Utilizing them, we construct new gauge invariant actions for heterotic NS string field theory. We derive the condition for the equivalence of the new action and the  $L_\infty$  action, which give different parameterizations of the same WZW-like structure and action, on the basis of the procedure demonstrated in [29]. Then we also derive the relation between two dynamical string field  $\Phi$  and  $V$  from the equivalence condition. We end with some conclusions and discussions.

Basic facts and definitions of the coalgebraic notation of  $A_\infty/L_\infty$  algebras are summarized in appendix A. In appendix B, we derive a formula which is used in section 2. In appendix C, we consider the trivially embedding of the string field of the  $L_\infty$  action belonging to the small Hilbert space into the string field in the large Hilbert space, and show the embedded action can also be written in the WZW-like form. Appendices D and E are devoted to the open NS and the closed NS-NS theories, respectively.

## 2 WZW-like structure from the $L_\infty$ -formulation

In this section, we clarify a WZW-like structure naturally arising from the  $L_\infty$ -formulation. After a brief review of the  $L_\infty$ -formulation [7], we show that the action can be written in the WZW-like form. We see that the functionals which appear in the action satisfy the  $\mathbb{Z}_2$ -reversed version of the conventional WZW-like relations in [19], which guarantees the gauge invariance of the WZW-like form of the  $L_\infty$ -action.

### Preliminaries

The product of  $n$  closed strings is described by a multilinear map  $b_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}$ , where



$\wedge$  is the *symmetrized tensor product* satisfying  $\Phi_1 \wedge \Phi_2 = (-)^{\deg(\Phi_1)\deg(\Phi_2)} \Phi_2 \wedge \Phi_1$ . A map  $b_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}$  with degree 1 naturally induces a map from the *symmetrized tensor algebra*  $\mathcal{S}(\mathcal{H}) = \mathcal{H}^{\wedge 0} \oplus \mathcal{H}^{\wedge 1} \oplus \mathcal{H}^{\wedge 2} \oplus \dots$  to  $\mathcal{S}(\mathcal{H})$  itself, called a *coderivation*. A map  $b_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}'$  with degree 0 also naturally induces a map from  $\mathcal{S}(\mathcal{H})$  to  $\mathcal{S}(\mathcal{H}')$ , called a *cohomomorphism*. Since it is convenient to write the functionals of the string field and the action in terms of them, we briefly introduce the rules of their actions here.

The coderivation  $\mathbf{d}_n : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H})$  is naturally derived from a map  $d_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}$  with degree one. It act on  $\Phi_1 \wedge \dots \wedge \Phi_N \in \mathcal{H}^{\wedge N \geq n} \subset \mathcal{S}(\mathcal{H})$  as

$$\begin{aligned} \mathbf{d}_n(\Phi_1 \wedge \dots \wedge \Phi_N) &= (d_n \wedge \mathbb{I}_{N-n})(\Phi_1 \wedge \dots \wedge \Phi_N) \\ &= \sum_{\sigma} \frac{(-)^{\sigma}}{n!(N-n)!} d_n(\Phi_{\sigma(1)}, \dots, \Phi_{\sigma(n)}) \wedge \Phi_{\sigma(n+1)} \wedge \dots \wedge \Phi_{\sigma(N)}, \end{aligned} \quad (2.1)$$

and vanishes when acting on  $\mathcal{H}^{\wedge N \leq n}$ . The graded commutator of two coderivations  $\mathbf{b}_n$  and  $\mathbf{c}_m$ ,  $[\mathbf{b}_n, \mathbf{c}_m]$ , is a coderivation derived from the map  $[[b_n, c_m]] : \mathcal{H}^{\wedge n+m-1} \rightarrow \mathcal{H}$  which is defined by

$$[[b_n, c_m]] = b_n(c_m \wedge \mathbb{I}_{n-1}) - (-)^{\deg(b_n)\deg(c_m)} c_m(b_n \wedge \mathbb{I}_{m-1}). \quad (2.2)$$

A set of degree zero multilinear maps  $\{f_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}'\}_{n=0}^{\infty}$  naturally induces a cohomomorphism  $\widehat{f} : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H}')$ , which acts on  $\Phi_1 \wedge \dots \wedge \Phi_n \in \mathcal{H}^{\wedge n} \subset \mathcal{S}(\mathcal{H})$  as

$$\begin{aligned} \widehat{f}(\Phi_1 \wedge \dots \wedge \Phi_n) &= \sum_{i \leq n} \sum_{k_1 < \dots < k_i} e^{\wedge f_0} \wedge f_{k_1}(\Phi_1, \dots, \Phi_{k_1}) \wedge f_{k_2-k_1}(\Phi_{k_1+1}, \dots, \Phi_{k_2}) \wedge \\ &\quad \dots \wedge f_{k_i-k_{i-1}}(\Phi_{k_{i-1}+1}, \dots, \Phi_n). \end{aligned} \quad (2.3)$$

For definitions and their more details, see appendix A.

## 2.1 Construction of $L_{\infty}$ -product and $L_{\infty}$ action

In this subsection, we briefly review the construction of the NS superstring product  $\mathbf{L}$  and the action  $S_{\text{EKS}}[\Phi]$  for heterotic string field theory in the  $L_{\infty}$  formulation originally given in [7].

### NS superstring product $\mathbf{L}$

Let us review the construction of the NS string products  $\mathbf{L}[\tau] = \sum_{p=0}^{\infty} \tau^p \mathbf{L}_{p+1}$  satisfying  $L_{\infty}$ -relation  $[[\mathbf{L}[\tau], \mathbf{L}[\tau]]] = 0$ , the cyclicity  $\mathbf{L}^{\dagger} = -\mathbf{L}$ , and the  $\eta$ -derivation  $[[\eta, \mathbf{L}[\tau]]] = 0$ . The  $(p+1)$ -product  $\mathbf{L}_{p+1}$  carries the ghost number  $1-2p$  and the picture number  $p$ . The products  $\mathbf{L}[\tau]$  consist of  $\eta$ ,  $\xi$ , and Zwiebach's bosonic string products  $\mathbf{L}^{\text{B}} = \mathbf{Q} + \mathbf{L}_2^{\text{B}} + \mathbf{L}_3^{\text{B}} + \dots$  satisfying  $[[\mathbf{L}^{\text{B}}, \mathbf{L}^{\text{B}}]] = 0$  [36]. First, we focus on the condition for  $\mathbf{L}[\tau]$  to satisfy the  $L_{\infty}$ -relations:

$$[[\mathbf{L}[\tau], \mathbf{L}[\tau]]] = 0. \quad (2.4)$$

The  $L_{\infty}$ -relations holds if we define  $\mathbf{L}[\tau]$  as a solution of the differential equations

$$\partial_{\tau} \mathbf{L}[\tau] = [[\mathbf{L}[\tau], \lambda^{[0]}[\tau]]], \quad (2.5)$$

with the initial condition  $\mathbf{L}[\tau=0] = \mathbf{Q}$ . Here  $\lambda^{[0]}[\tau] = \sum_{p=0}^{\infty} \tau^p \lambda_{p+2}^{[0]}$  are called gauge products. We can take any  $\lambda^{[0]}$  as long as they carry correct quantum numbers: the  $(p+2)$ -product  $\lambda_{p+2}^{[0]}$

carries ghost number  $-2(p+1)$  and picture number  $p+1$ . The solution for the differential equations is given by the similarity transformation of  $Q$ :

$$\mathbf{L}_{[\tau]} = \widehat{\mathbf{G}}^{-1}_{[\tau]} \mathbf{Q} \widehat{\mathbf{G}}_{[\tau]}, \quad (2.6)$$

where  $\widehat{\mathbf{G}}$  is an invertible cohomomorphism which is given by the path-ordered exponential of the gauge products  $\lambda^{[0]}_{[\tau]}$ :

$$\widehat{\mathbf{G}}_{[\tau]} = \overleftarrow{\mathcal{P}} \exp \left( \int_0^\tau d\tau' \lambda^{[0]}_{[\tau']} \right), \quad \widehat{\mathbf{G}}^{-1}_{[\tau]} = \overrightarrow{\mathcal{P}} \exp \left( - \int_0^\tau d\tau' \lambda^{[0]}_{[\tau']} \right), \quad (2.7)$$

where  $\leftarrow(\rightarrow)$  on  $\mathcal{P}$  denotes that the operator at later time acts from the right (left). The differential equations hold since the path-ordered exponentials satisfy

$$\partial_\tau \widehat{\mathbf{G}}_{[\tau]} = \widehat{\mathbf{G}}_{[\tau]} \lambda^{[0]}_{[\tau]}, \quad \partial_\tau \widehat{\mathbf{G}}^{-1}_{[\tau]} = -\lambda^{[0]}_{[\tau]} \widehat{\mathbf{G}}^{-1}_{[\tau]}. \quad (2.8)$$

We may check directly the  $L_\infty$ -relations as follows:

$$\mathbf{L}^2 = \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}} \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}} = \widehat{\mathbf{G}}^{-1} \mathbf{Q} \mathbf{Q} \widehat{\mathbf{G}} = 0. \quad (2.9)$$

The cyclicity and the  $\eta$ -derivation properties follow from the suitable choice of  $\lambda^{[0]}$ . Second, let us consider the cyclicity. For  $\mathbf{L}$  to be cyclic, it is sufficient to choose  $\lambda^{[0]}$  to be BPZ-odd, so that  $\widehat{\mathbf{G}}^{-1} = \widehat{\mathbf{G}}^\dagger$ :

$$\mathbf{L}^\dagger = (\widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}})^\dagger = -\widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}} = -\mathbf{L}. \quad (2.10)$$

Third, we check  $\eta$  acts as a derivation on  $\mathbf{L}$ , namely the  $\eta$ -derivation properties

$$[\![\eta, \mathbf{L}_{[\tau]}]\!] = 0. \quad (2.11)$$

A construction of the suitable gauge product  $\lambda^{[0]}$  is given in [7]. It is helpful to consider a series of generating functions

$$\mathbf{L}(s, \tau) = \sum_{d=0}^{\infty} s^d \mathbf{L}^{[d]}_{[\tau]} = \sum_{d=0}^{\infty} \sum_{p=0}^{\infty} s^d \tau^p \mathbf{L}^{[d]}_{p+d+1}, \quad (2.12)$$

including  $\mathbf{L}_{[\tau]} = \mathbf{L}^{[0]}_{[\tau]}$  and  $\mathbf{L}_N^{\mathbf{B}} = \mathbf{L}_N^{[N-1]}$ , where the superscript  $[d]$  denotes the picture deficit relative to what is needed for the NS products. They satisfy the  $L_\infty$ -relations and  $\eta$ -derivation properties:

$$[\![\mathbf{L}(s, \tau), \mathbf{L}(s, \tau)]\!] = 0, \quad (2.13)$$

$$[\![\eta, \mathbf{L}(s, \tau)]\!] = 0, \quad (2.14)$$

if we define  $\mathbf{L}(s, \tau)$  as a solution for the following differential equations:

$$\partial_\tau \mathbf{L}(s, \tau) = [\![\mathbf{L}(s, \tau), \lambda(s, \tau)]\!], \quad (2.15)$$

$$\partial_s \mathbf{L}(s, \tau) = [\![\eta, \lambda(s, \tau)]\!], \quad (2.16)$$

where  $\lambda(s, \tau)$  is a series of generating functions for the gauge products

$$\lambda(s, \tau) = \sum_{d=0}^{\infty} s^d \lambda^{[d]}_{[\tau]} = \sum_{d=0}^{\infty} \sum_{p=0}^{\infty} s^d \tau^p \lambda^{[d]}_{d+p+2}, \quad (2.17)$$

including  $\lambda_{[\tau]} = \lambda^{[0]}_{[\tau]}$ . The expansion of the differential equations (2.15) and (2.16) in powers of  $s$  and  $\tau$  provides the recursive definition of  $\mathbf{L}_N^{[d]}$  and  $\lambda_N^{[d]}$ . Note that the explicit forms of  $\lambda_N^{[d]}$  is not determined uniquely even if we require all the desired properties, which follows from the arbitrariness of the inverse of  $\eta$  in (2.16). Then we obtain the suitable  $\lambda^{[0]}_{[\tau]}$  which provides  $\eta$ -derivation properties of  $\mathbf{L}_{[\tau]} = \mathbf{L}^{[0]}_{[\tau]}$ , the  $s^0$  part of (2.14). Hereafter we take  $\tau = 1$  and omit the argument:

$$\mathbf{L} := \mathbf{L}_{[\tau=1]}. \quad (2.18)$$

### Action for heterotic string field theory in the $L_\infty$ -formulation

The gauge-invariant action in the  $L_\infty$ -formulation is constructed by [7], using the above products  $\mathbf{L} = \{L_k\}_{k \geq 1}$ . The dynamical string field  $\Phi$  in the  $L_\infty$ -formulation carries ghost number 2 and picture number  $-1$  and belongs to the small Hilbert space  $\mathcal{H}_{\text{small}}$ :  $\eta\Phi = 0$ . The action is written as follows:

$$S_{\text{EKS}}[\Phi] = \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \langle \xi\Phi, L_{n+1}(\overbrace{\Phi, \Phi, \dots, \Phi}^{n+1}) \rangle, \quad (2.19)$$

where  $L_1 = Q$  and  $\langle A, B \rangle$  is the BPZ inner product with the  $c_0^- = \frac{1}{2}(c_0 - \tilde{c}_0)$ -insertion.

Let us introduce the  $t$ -parametrized string field  $\Phi(t)$  with  $t \in [0, 1]$  satisfying  $\Phi(0) = 0$  and  $\Phi(1) = \Phi$ , which is a path connecting 0 and the string field  $\Phi$  in the space of string fields. Utilizing this  $\Phi(t)$ , the action can be represented in the following form:

$$\begin{aligned} S_{\text{EKS}}[\Phi] &= \int_0^1 dt \frac{\partial}{\partial t} \left( \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \langle \xi\Phi(t), L_{n+1}(\overbrace{\Phi(t), \dots, \Phi(t)}^{n+1}) \rangle \right) \\ &= \int_0^1 dt \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \langle \partial_t \xi\Phi(t), L_{n+1}(\overbrace{\Phi(t), \dots, \Phi(t)}^{n+1}) \rangle. \end{aligned} \quad (2.20)$$

Here we use  $\Phi(t) = \eta\xi\Phi(t)$  and  $\eta$ -derivation property and the cyclicity of  $\mathbf{L}$ , in order to obtain the form in which both  $\partial_t$  and  $\xi$  act on the first slot of the inner product. Note that this  $t$ -dependence is topological, since it does not appear in the variation of the action, as we will see later. We can represent the action in the coalgebraic notation as follows:

$$S_{\text{EKS}}[\Phi] = \int_0^1 dt \langle \pi_1(\xi_t e^{\wedge\Phi(t)}), \pi_1(\mathbf{L}(e^{\wedge\Phi(t)})) \rangle, \quad (2.21)$$

where  $e^{\wedge\Phi}$  is the *group-like element* defined by

$$e^{\wedge\Phi} = \mathbf{1} + \Phi + \frac{1}{2}\Phi \wedge \Phi + \frac{1}{3!}\Phi \wedge \Phi \wedge \Phi + \dots, \quad (2.22)$$

$\pi_1$  is a projector from the symmetrized tensor algebra to the single-state space

$$\pi_1(\Phi_0 + \Phi_1 \wedge \Phi_2 + \Phi_3 \wedge \Phi_4 \wedge \Phi_5 + \dots) = \Phi_0, \quad (2.23)$$

and  $\xi_t$  is a coderivation<sup>5</sup> derived from the linear map  $\xi \partial_t : \Phi \mapsto \xi \partial_t \Phi$ . See also appendix ??.

The variation of the action can be taken as follows:

$$\delta S_{\text{EKS}}[\Phi] = \langle \xi \delta \Phi, \pi_1(\mathbf{L}(e^{\wedge \Phi})) \rangle, \quad (2.24)$$

and then the equation of motion is given by

$$\pi_1 \mathbf{L}(e^{\wedge \Phi}) = 0. \quad (2.25)$$

Since  $\mathbf{L}^2 = 0$ , the action is invariant under the following gauge transformation:

$$\delta \Phi = \pi_1 \mathbf{L}(\lambda \wedge e^{\wedge \Phi}), \quad (2.26)$$

where  $\lambda$  is the gauge parameter in the small Hilbert space carrying ghost number 1 and picture number  $-1$ .

## 2.2 $L_\infty$ action in the WZW-like form

The action in the  $L_\infty$  formulation can be transformed as follows:

$$\begin{aligned} S_{\text{EKS}}[\Phi] &= \int_0^1 dt \langle \pi_1(\xi_t e^{\wedge \Phi(t)}), \pi_1(\widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}(e^{\wedge \Phi(t)})) \rangle \\ &= \int_0^1 dt \langle \pi_1(\widehat{\mathbf{G}}(\xi_t e^{\wedge \Phi(t)})), \pi_1 \mathbf{Q}(\widehat{\mathbf{G}}(e^{\wedge \Phi(t)})) \rangle. \end{aligned} \quad (2.27)$$

See [32], and [29] for open string. The functionals of the dynamical string field  $\Phi$  appearing in the action

$$\Psi_\eta[\Phi(t)] = \pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}), \quad (2.28)$$

$$\Psi_d[\Phi(t)] = \pi_1 \widehat{\mathbf{G}}(\xi_d e^{\wedge \Phi(t)}), \quad (2.29)$$

play important roles. One can show that, introducing a certain set of products satisfying  $L_\infty$ -relations

$$\eta, \quad [\cdot, \cdot]^\eta, \quad [\cdot, \cdot, \cdot]^\eta, \quad \dots, \quad (2.30)$$

the functionals  $\Psi_\eta[\Phi(t)]$  and  $\Psi_d[\Phi(t)]$  satisfy the following relations:

$$0 = \eta \Psi_\eta + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \overbrace{[\Psi_\eta, \dots, \Psi_\eta]}^{n+1}{}^\eta, \quad (2.31)$$

$$(-)^d d \Psi_\eta = \eta \Psi_d + \sum_{k=1}^{\infty} \frac{1}{k!} \overbrace{[\Psi_\eta, \dots, \Psi_\eta, \Psi_d]}^k{}^\eta, \quad (2.32)$$

which are  $\mathbb{Z}_2$ -reversed versions of WZW-like relations in [19]. In this subsection, after defining these  $L_\infty$ -products (2.30), which we call the dual products, we confirm a pair of fields (2.28) and (2.29) satisfy WZW-like relations.

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<sup>5</sup>We will define  $\xi_d$  for the more general class of  $d$  later. For  $\partial_t$ , definitions are equivalent.

### Construction of the dual $L_\infty$ -product $\mathbf{L}^\eta$

The dual  $L_\infty$ -products  $\mathbf{L}^\eta$  can be constructed using the cohomomorphism  $\widehat{\mathbf{G}}$  which provides the NS heterotic string products  $\mathbf{L} = \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}$ . The product  $\mathbf{L}^\eta$  is defined as the similarity transformation of  $\eta$ :

$$\mathbf{L}^\eta = \widehat{\mathbf{G}} \eta \widehat{\mathbf{G}}^{-1} = \sum_{n=1}^{\infty} \mathbf{L}_n^\eta. \quad (2.33)$$

These  $\mathbf{L}^\eta$  carry odd degree, and the  $n$ -product  $\mathbf{L}_n^\eta$  carries ghost number  $3 - 2n$  and picture number  $n - 2$ .  $\mathbf{L}^\eta$  satisfy the  $L_\infty$ -relations, which follow from its definition:

$$(\mathbf{L}^\eta)^2 = \widehat{\mathbf{G}} \eta \widehat{\mathbf{G}}^{-1} \widehat{\mathbf{G}} \eta \widehat{\mathbf{G}}^{-1} = \widehat{\mathbf{G}} \eta \eta \widehat{\mathbf{G}}^{-1} = 0. \quad (2.34)$$

The  $Q$ -derivation properties of  $\mathbf{L}^\eta$  follow from  $\eta$ -derivatioies property of  $\mathbf{L}$ :

$$\mathbf{Q} \mathbf{L}^\eta = \widehat{\mathbf{G}} (\widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}) \eta \widehat{\mathbf{G}}^{-1} = -\widehat{\mathbf{G}} \eta (\widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}) \widehat{\mathbf{G}}^{-1} = -\mathbf{L}^\eta \mathbf{Q}. \quad (2.35)$$

The cyclicity of  $L^\eta$  follows from that the gauge products are BPZ-odd:

$$(\mathbf{L}^\eta)^\dagger = (\widehat{\mathbf{G}} \eta \widehat{\mathbf{G}}^{-1})^\dagger = (\widehat{\mathbf{G}}^{-1})^\dagger \eta^\dagger \widehat{\mathbf{G}}^\dagger = -\widehat{\mathbf{G}} \eta \widehat{\mathbf{G}}^{-1}. \quad (2.36)$$

The dual  $L_\infty$ -products  $\mathbf{L}^\eta$  is the generating function for (2.30):

$$[B_1, \dots, B_n]^\eta := \pi_1 \mathbf{L}_n^\eta(B_1 \wedge \dots \wedge B_n). \quad (2.37)$$

The dual products satisfy the  $L_\infty$ -relations,  $Q$ -derivation properties, and cyclicity:

$$\sum_{\sigma} \sum_{k=1}^n \frac{1}{k!(n-k)!} (-)^{|\sigma|} [[B_{i_{\sigma(1)}}, \dots, B_{i_{\sigma(k)}}]^\eta, B_{i_{\sigma(k+1)}}, \dots, B_{i_{\sigma(n)}}]^\eta = 0, \quad (2.38)$$

$$Q[B_1, \dots, B_n]^\eta + \sum_{i=1}^n (-)^{B_1 + \dots + B_{i-1}} [B_1, \dots, Q B_i, \dots, B_n]^\eta = 0, \quad (2.39)$$

$$\langle B_1, [B_2, \dots, B_{n+1}]^\eta \rangle = (-)^{B_1 + B_2 + \dots + B_n} \langle [B_1, \dots, B_n]^\eta, B_{n+1} \rangle, \quad (2.40)$$

where  $(-)^{\sigma}$  is the sign factor of the permutation  $\{A_{\sigma(1)}, \dots, A_{\sigma(n)}\}$ .

The shifted products of  $\mathbf{L}^\eta$  are defined by

$$[B_1, B_2, \dots, B_n]_A^\eta = \sum_{m=0}^{\infty} \frac{1}{m!} [\overbrace{A, A, \dots, A}^m, B_1, B_2, \dots, B_n]^\eta. \quad (2.41)$$

They satisfy  $L_\infty$ -relations if  $A$  is a solution for the Maurer-Cartan equation of  $\mathbf{L}^\eta$ :

$$0 = \pi_1 \mathbf{L}^\eta(e^{\wedge A}) = \eta A + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} [\overbrace{A, \dots, A}^{n+1}]^\eta. \quad (2.42)$$

Let  $\Psi_\eta$  be a solution for the Maurer-Cartan equation of  $\mathbf{L}^\eta$ . In particular we write  $D_\eta$  for the  $\Psi_\eta$ -shifted 1-product:

$$D_\eta B := [B]_{\Psi_\eta}^\eta = \sum_{m=0}^{\infty} \frac{1}{m!} [\overbrace{\Psi_\eta, \Psi_\eta, \dots, \Psi_\eta}^m, B]^\eta. \quad (2.43)$$

From the  $L_\infty$ -relation of the  $\Psi_\eta$ -shifted  $\mathbf{L}^\eta$ ,  $D_\eta$  is nilpotent:

$$(D_\eta)^2 B = -[\pi_1 \mathbf{L}^\eta(e^{\wedge \Psi_\eta}), B]_{\Psi_\eta}^\eta = 0, \quad (2.44)$$

and acts on the  $\Psi_\eta$ -shifted 2-product  $[B_1, B_2]_{\Psi_\eta}^\eta$  as a derivation:

$$D_\eta[B_1, B_2]_{\Psi_\eta}^\eta + [D_\eta B_1, B_2]_{\Psi_\eta}^\eta + (-)^{B_1}[B_1, D_\eta B_2]_{\Psi_\eta}^\eta = -[\pi_1 \mathbf{L}^\eta(e^{\wedge \Psi_\eta}), B_1, B_2]_{\Psi_\eta}^\eta = 0 \quad (2.45)$$

The shifted products are BPS odd, which follows from that of  $\mathbf{L}^\eta$ :

$$\langle B_1, [B_2, \dots, B_{n+1}]_{\Psi_\eta}^\eta \rangle = (-)^{B_1+B_2+\dots+B_n} \langle [B_1, \dots, B_n]_{\Psi_\eta}^\eta, B_{n+1} \rangle. \quad (2.46)$$

### WZW-like relations

Let us confirm a pair of fields (2.28) and (2.29) satisfy the WZW-like relations, which can be represented as follows:

$$0 = \pi_1 \mathbf{L}^\eta(e^{\wedge \Psi_\eta}), \quad (2.47)$$

$$(-)^d d \Psi_\eta = D_\eta \Psi_d, \quad (2.48)$$

The first relation (2.47) directly follows from the fact that  $\Phi$  belongs to small space:

$$\mathbf{L}^\eta(e^{\wedge \Psi_\eta[\Phi(t)]}) = \mathbf{L}^\eta(e^{\wedge \pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)})}) = (\widehat{\mathbf{G}} \boldsymbol{\eta} \widehat{\mathbf{G}}^{-1}) \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}) = \widehat{\mathbf{G}} \boldsymbol{\eta}(e^{\wedge \Phi(t)}) = 0. \quad (2.49)$$

We call  $\Psi_\eta(t)$  satisfying (2.47) the *pure-gauge-like field*.

The second relation (2.48) can be confirmed similarly. The operator  $\mathbf{d}$  which we focus on is the derivation on  $\mathbf{L}^\eta$ . Their derivation property on  $\mathbf{L}^\eta$  lead to  $[\widehat{\mathbf{G}}^{-1} \mathbf{d} \widehat{\mathbf{G}}, \boldsymbol{\eta}] = 0$ , and we can define the coderivation  $\boldsymbol{\xi}_d$  such that

$$\widehat{\mathbf{G}}^{-1} \mathbf{d} \widehat{\mathbf{G}} = (-)^d [\boldsymbol{\eta}, \boldsymbol{\xi}_d]. \quad (2.50)$$

Note that for the operator  $\mathbf{d}$  which commutes with  $\widehat{\mathbf{G}}$ , such as  $\partial_t$  and  $\delta$ ,  $\boldsymbol{\xi}_d$  is a coderivation derived from  $\mathbf{d}\boldsymbol{\xi}$ . Then, utilizing this  $\boldsymbol{\xi}_d$ , the following relation holds:

$$\begin{aligned} (-)^d \mathbf{d} \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}) &= (-)^d \widehat{\mathbf{G}}(\widehat{\mathbf{G}}^{-1} \mathbf{d} \widehat{\mathbf{G}})(e^{\wedge \Phi(t)}) \\ &= \widehat{\mathbf{G}} \boldsymbol{\eta} \boldsymbol{\xi}_d(e^{\wedge \Phi(t)}) \\ &= \mathbf{L}^\eta \widehat{\mathbf{G}} \boldsymbol{\xi}_d(e^{\wedge \Phi(t)}) \\ &= \mathbf{L}^\eta \left( \pi_1 \widehat{\mathbf{G}} \boldsymbol{\xi}_d(e^{\wedge \Phi(t)}) \wedge e^{\wedge \pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)})} \right). \end{aligned} \quad (2.51)$$

Since  $D_\eta = \pi_1 \mathbf{L}^\eta(\mathbb{I} \wedge e^{\wedge \Psi_\eta})$ , we can see  $\Psi_\eta[\Phi(t)]$  and  $\Psi_d[\Phi(t)]$  satisfy the WZW-like relation (2.48) using (??), (2.29), (2.50) and (2.51). We call  $\Psi_d(t)$  satisfying (2.48) the *associated field*.

Then the action in the  $L_\infty$ -formulation can be written in the WZW-like form:

$$S_{\text{EKS}}[\Phi] = \int_0^1 dt \langle \Psi_t[\Phi(t)], Q \Psi_\eta[\Phi(t)] \rangle. \quad (2.52)$$

The variation of the action can be taken easily using the WZW-like relations (2.47) and (2.48), and the gauge invariances also follows from them, which can be seen in a similar (but  $\mathbb{Z}_2$ -reversed) manner to those in [19]. We will see them in the next subsection.

### 2.3 Variation of the action

Let us take the variation of the action. Note that in the computation here we use only the WZW-like relations (2.47) and (2.48) and not the explicit forms of the functionals  $\Psi_\eta$  and  $\Psi_d$ . Therefore, even if the functionals  $\Psi_\eta$  and  $\Psi_d$  have different forms, the variation of the action can be taken in the same manner as long as they satisfy the WZW-like relations.

First, consider the variation of the integrand of (2.52):

$$\delta \langle \Psi_t(t), Q\Psi_\eta(t) \rangle = \langle \delta\Psi_t(t), Q\Psi_\eta(t) \rangle + \langle \Psi_t(t), Q\delta\Psi_\eta(t) \rangle. \quad (2.53)$$

Utilizing the following relation which follows from (2.48):

$$0 = \llbracket d_1, d_2 \rrbracket \Psi_\eta = (-)^{d_1+d_2} D_\eta (d_1 \Psi_{d_2} - (-)^{d_1 d_2} d_2 \Psi_{d_1} + (-)^{d_2 d_1 + d_2} [\Psi_{d_2}, \Psi_{d_1}]_\Psi^\eta), \quad (2.54)$$

the first term can be transformed into

$$\langle \delta\Psi_t(t), Q\Psi_\eta(t) \rangle = \langle \partial_t \Psi_\delta(t) + [\Psi_\delta(t), \Psi_t(t)]_{\Psi_\eta(t)}^\eta, Q\Psi_\eta(t) \rangle. \quad (2.55)$$

Utilizing  $\llbracket D_\eta, d \rrbracket B = -[d\Psi_\eta, B]_{\Psi_\eta}^\eta$ , the second term can be transformed into

$$\begin{aligned} \langle \Psi_t(t), Q\delta\Psi_\eta(t) \rangle &= \langle \Psi_t(t), QD_\eta \Psi_\delta(t) \rangle \\ &= \langle D_\eta Q\Psi_t(t), \Psi_\delta(t) \rangle \\ &= \langle -QD_\eta \Psi_t(t) - [Q\Psi_\eta(t), \Psi_t(t)]_{\Psi_\eta(t)}^\eta, \Psi_\delta(t) \rangle. \end{aligned} \quad (2.56)$$

The second terms of (2.55) and (2.56) are canceled because of the cyclicity of  $\Psi_\eta$ -shifted  $\mathbf{L}^\eta$ . Then we find that the variation of the integrand of (2.52) becomes a total derivative of  $t$ :

$$\begin{aligned} \delta \langle \Psi_t(t), Q\Psi_\eta(t) \rangle &= \langle \partial_t \Psi_\delta(t), Q\Psi_\eta(t) \rangle - \langle QD_\eta \Psi_t(t), \Psi_\delta(t) \rangle \\ &= \langle \partial_t \Psi_\delta(t), Q\Psi_\eta(t) \rangle + \langle \Psi_\delta(t), Q\partial_t \Psi_\eta(t) \rangle \\ &= \partial_t \langle \Psi_\delta(t), Q\Psi_\eta(t) \rangle. \end{aligned} \quad (2.57)$$

Integrating over  $t$ , the variation of the action is given by

$$\int_0^1 dt \delta \langle \Psi_t(t), Q\Psi_\eta(t) \rangle = \int_0^1 dt \partial_t \langle \Psi_\delta(t), Q\Psi_\eta(t) \rangle = \langle \Psi_\delta(1), Q\Psi_\eta(1) \rangle, \quad (2.58)$$

where the pure-gauge-like field  $\Psi_\eta$  and the associated field  $\Psi_d$  vanish at  $t = 0$ . Then the variation of the action becomes

$$\delta S_{\text{EKS}} = \langle \Psi_\delta[\Phi], Q\Psi_\eta[\Phi] \rangle. \quad (2.59)$$

We find that the variation of the action does not depend on  $t$ , and therefore  $t$ -dependence is topological.

#### Equations of motion

One can derive the on-shell condition from the variation of the action in the WZW-like form (2.59),

$$Q\Psi_\eta[\Phi] = 0. \quad (2.60)$$

For completion, let us discuss the equivalence of (2.60) and (2.25).

The latter (2.25) can be transformed into the following form:

$$\pi_1 \mathbf{L}(e^{\wedge \Phi}) = \pi_1 \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}(e^{\wedge \Phi}) = \pi_1 \widehat{\mathbf{G}}^{-1} \mathbf{Q}(e^{\wedge \Psi_\eta[\Phi]}) = \pi_1 \widehat{\mathbf{G}}^{-1}((Q\Psi_\eta[\Phi]) \wedge e^{\wedge \Psi_\eta[\Phi]}). \quad (2.61)$$

Since  $\pi_1 \widehat{\mathbf{G}}^{-1}(\cdot \wedge e^{\wedge \Psi_\eta})$  is invertible, (2.60) and (2.25) are equivalent:

$$\pi_1 \mathbf{L}(e^{\wedge \Phi}) = 0 \iff Q\Psi_\eta[\Phi] = 0. \quad (2.62)$$

Note that the overall factor  $\pi_1 \widehat{\mathbf{G}}^{-1}(\cdot \wedge e^{\wedge \Psi_\eta})$  comes from the difference of the  $\xi\delta\Phi$  and  $\Psi_\delta$ .

## 2.4 Gauge transformations

It follows from the nilpotency of  $Q$  and  $D_\eta$  that the WZW-like action is invariant under the following form of the gauge transformations

$$\Psi_\delta = D_\eta \Omega + Q\Lambda, \quad (2.63)$$

where  $\Omega$  and  $\Lambda$  are the gauge parameters belonging to the large Hilbert space, which carry ghost numbers 0 and 0, and picture numbers 1 and 0, respectively. In this subsection, we see how the gauge transformation (2.26) can be represented in the WZW-like form (2.63).

Let us consider the associated field  $\Psi_\delta[\Phi] = \pi_1 \widehat{\mathbf{G}}(\xi_\delta e^{\wedge \Phi}) = \pi_1 \widehat{\mathbf{G}}((\xi\delta\Phi) \wedge e^{\wedge \Phi})$ , with  $\delta\Phi$  being the gauge transformation (2.26) in the  $L_\infty$ -formulation:

$$\delta\Phi = \pi_1 \mathbf{L}(\lambda \wedge e^{\wedge \Phi}) = \pi_1 \mathbf{L}\eta(\xi\lambda \wedge e^{\wedge \Phi}) = -\eta\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge \Phi}). \quad (2.64)$$

$\Psi_\delta[\Phi]$  can be transformed as follows:

$$\begin{aligned} \Psi_\delta[\Phi] &= \pi_1 \widehat{\mathbf{G}}\left((- \xi\eta\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge \Phi})) \wedge e^{\wedge \Phi}\right) \\ &= \pi_1 \widehat{\mathbf{G}}\left((\eta\xi\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge \Phi})) \wedge e^{\wedge \Phi}\right) - \pi_1 \widehat{\mathbf{G}}\left((\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge \Phi})) \wedge e^{\wedge \Phi}\right) \\ &= \pi_1 \widehat{\mathbf{G}}\eta\left((\xi\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge \Phi})) \wedge e^{\wedge \Phi}\right) - \pi_1 \widehat{\mathbf{G}}\left(\mathbf{L}(\xi\lambda \wedge e^{\wedge \Phi}) + \xi\lambda \wedge \pi_1 \mathbf{L}(e^{\wedge \Phi}) \wedge e^{\wedge \Phi}\right) \\ &= \pi_1 \mathbf{L}^\eta \widehat{\mathbf{G}}\left((\xi\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge \Phi})) \wedge e^{\wedge \Phi}\right) - \pi_1 \mathbf{Q} \widehat{\mathbf{G}}(\xi\lambda \wedge e^{\wedge \Phi}) - \pi_1 \widehat{\mathbf{G}}(\xi\lambda \wedge \pi_1 \mathbf{L}(e^{\wedge \Phi}) \wedge e^{\wedge \Phi}) \\ &= \pi_1 \mathbf{L}^\eta\left(\pi_1 \widehat{\mathbf{G}}((\xi\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge \Phi})) \wedge e^{\wedge \Phi}) \wedge e^{\wedge \pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi})}\right) - Q\pi_1 \widehat{\mathbf{G}}(\xi\lambda \wedge e^{\wedge \Phi}) - \Delta_T[\lambda, \Phi] \\ &= D_\eta \Omega[\lambda, \Phi] + Q\Lambda[\lambda, \Phi] - \Delta_T[\lambda, \Phi], \end{aligned} \quad (2.65)$$

The first two terms of (2.65) correspond to (2.63), with the gauge parameters  $\Omega = \Omega[\lambda, \Phi]$  and  $\Lambda = \Lambda[\lambda, \Phi]$  parameterized by  $\lambda$  as follows:

$$\Omega[\lambda, \Phi] = \pi_1 \widehat{\mathbf{G}}((\xi\pi_1 \mathbf{L}(\xi\lambda \wedge e^{\wedge \Phi})) \wedge e^{\wedge \Phi}), \quad (2.66)$$

$$\Lambda[\lambda, \Phi] = -\pi_1 \widehat{\mathbf{G}}(\xi\lambda \wedge e^{\wedge \Phi}), \quad (2.67)$$

while the third term of (2.65) ,

$$\Delta_T[\lambda, \Phi] = \pi_1 \widehat{\mathbf{G}}(\xi\lambda \wedge \pi_1 \mathbf{L}(e^{\wedge \Phi}) \wedge e^{\wedge \Phi}), \quad (2.68)$$

corresponds to the trivial gauge transformation of the WZW-like action.

Thus, the gauge transformation in the  $L_\infty$ -formulation (2.26) can be written in the WZW-like form (2.63): while the gauge invariance under (2.26) follows from the  $L_\infty$ -relations of  $\mathbf{L}$ , that under (2.63) follows from the WZW-like relations (2.47) and (2.48).



### Trivial gauge transformation

Trivial gauge transformation is a transformation proportional to the equations of motion. Schematically, it is of the following form:

$$\delta_\mu \varphi^i = (\text{EOM})_j \mu^{ji}, \quad \mu^{ji} = -(-)^{ij} \mu^{ij}, \quad (2.69)$$

and its gauge invariance follows from the symmetric property of  $\mu^{ji}$ :

$$\delta_\mu S = (\text{EOM})_i \delta_\mu \varphi^i = (\text{EOM})_i (\text{EOM})_j \mu^{ji} = 0. \quad (2.70)$$

They are no physical significance, but in general they may appear in the algebra of the nontrivial gauge transformations, and in the context of the Batalin-Vilkovisky quantization [37, 38] it is convenient to consider them<sup>6</sup>.

In our case, by the almost same computation with [31], when  $A + B + C = \text{even}$ , cyclicity of the cohomomorphism  $\widehat{H}$  is written as follows:

$$\langle \pi_1 \widehat{H}(A \wedge B \wedge e^{\wedge \Phi}), \pi_1 \widehat{H}(C \wedge e^{\wedge \Phi}) \rangle = -(-)^B \langle \pi_1 \widehat{H}(A \wedge e^{\wedge \Phi}), \pi_1 \widehat{H}(B \wedge C \wedge e^{\wedge \Phi}) \rangle. \quad (2.71)$$

The derivation is in appendix B. Take  $\widehat{H} = \widehat{\mathbf{G}}$ ,  $A = \xi\lambda$  which is even, and  $B = C = \pi_1 \mathbf{L} e^{\wedge \Phi}$  which are odd. The invariance under the trivial gauge transformation (2.68)<sup>7</sup>,

$$\begin{aligned} \langle -\Delta_T[\lambda, \Phi], Q\Psi_\eta[\Phi] \rangle &= \langle -\pi_1 \widehat{\mathbf{G}}(\xi\lambda \wedge \pi_1 \mathbf{L}(e^{\wedge \Phi}) \wedge e^{\wedge \Phi}), \pi_1 \widehat{\mathbf{G}}(\pi_1 \mathbf{L}(e^{\wedge \Phi}) \wedge e^{\wedge \Phi}) \rangle \\ &= -\langle \pi_1 \widehat{\mathbf{G}}(\xi\lambda \wedge e^{\wedge \Phi}), \pi_1 \widehat{\mathbf{G}}(\pi_1 \mathbf{L}(e^{\wedge \Phi}) \wedge \pi_1 \mathbf{L}(e^{\wedge \Phi}) \wedge e^{\wedge \Phi}) \rangle \\ &= 0 \end{aligned} \quad (2.72)$$

follows from the symmetric property:

$$\pi_1 \mathbf{L} e^{\wedge \Phi} \wedge \pi_1 \mathbf{L} e^{\wedge \Phi} = 0. \quad (2.73)$$

## 3 Alternative parameterization for NS closed string field theory

In section 2.3, we saw that, once the functionals  $\Psi_\eta$  and  $\Psi_d$  satisfying the WZW-like relations (2.47) and (2.48)

$$0 = \pi_1 \mathbf{L}^\eta(e^{\wedge \Psi_\eta}), \quad (-)^d d\Psi_\eta = D_\eta \Psi_d \quad (3.1)$$

are obtained, the gauge invariant action can be constructed as follows:

$$S_\eta = \int_0^1 dt \langle \Psi_t(t), Q\Psi_\eta(t) \rangle. \quad (3.2)$$

This form of the action is the  $\mathbb{Z}_2$ -reversed version of that in [19]. The  $L_\infty$ -action is one parameterization of this WZW-like action:

$$S_\eta = S_\eta[\Phi] := S_{\text{EKS}}[\Phi]. \quad (3.3)$$

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<sup>6</sup>For more detail, see [39].

<sup>7</sup>Note that, although the expression does not contain explicit WZW-like on-shell condition  $Q\Psi_\eta$ , the on-shell equivalence (2.62) guarantees it is trivial gauge transformation.

In this section we provide the alternative realization of these functionals, which is parameterized by the string field  $V$  in the large Hilbert space. We first see the pure-gauge-like and the associated string fields can be defined by the differential equations, which are the  $\mathbb{Z}_2$ -reversed versions of the construction in [19]. Then, the WZW-like action parameterized by the string field  $V$

$$S_\eta = S_\eta[V] \quad (3.4)$$

is given in terms of them. The equivalence of the actions in the different parameterizations is shown by the almost same procedure performed in [29].

### 3.1 Alternative parameterization by string field in large space

Let  $V$  be a dynamical string field which belongs to the large Hilbert space and carries ghost number 1 and picture number 0. In this subsection, we provide another parameterization of the pure-gauge-like field  $\Psi_\eta = \Psi_\eta[V]$  and the associated fields  $\Psi_d = \Psi_d[V]$ . A set of differential equations which are the  $\mathbb{Z}_2$ -reversed version of those in [19] give these parameterizations so that WZW-like relations (2.47) and (2.48) hold. Utilizing these functionals, the new gauge invariant action for  $V$  is constructed in the WZW-like form.

#### Pure-gauge-like field $\Psi_\eta = \Psi_\eta[V]$

A pure-gauge-like field of  $0 = \pi_1 \mathbf{L}^\eta(e^{\wedge \Psi_\eta})$  is the solution of the Maurer-Cartan equation for  $\mathbf{L}^\eta$ .  $\Psi_\eta[V]$  is obtained by mimicking the pure gauge construction of [19]. First, we introduce a real parameter  $\tau \in [0, 1]$ . Second, we solve the differential equation

$$\begin{aligned} \frac{\partial}{\partial \tau} \Psi_\eta[\tau; V] &= D_\eta(\tau)V \\ &= \eta V + \sum_{k=1}^{\infty} \frac{1}{k!} [\overbrace{\Psi_\eta[\tau; V], \dots, \Psi_\eta[\tau; V]}^k, V]^\eta, \end{aligned} \quad (3.5)$$

with the initial condition

$$\Psi_\eta[\tau = 0; V] = 0. \quad (3.6)$$

Finally, we set  $\tau = 1$  and obtain  $\Psi_\eta[V]$

$$\Psi_\eta[V] := \Psi_\eta[\tau = 1; V]. \quad (3.7)$$

One can check that the differential equation actually provides a solution for the Maurer-Cartan equation

$$\mathcal{F}(\Psi_\eta[\tau; V]) = \mathbf{L}^\eta e^{\wedge \Psi_\eta[\tau; V]} = 0, \quad (3.8)$$

by differentiating it by  $\tau$ :

$$\partial_\tau \mathcal{F}(\Psi_\eta) = \pi_1 \mathbf{L}^\eta (\partial_\tau \Psi_\eta \wedge e^{\wedge \Psi_\eta[\tau]}) = D_\eta(\partial_\tau \Psi_\eta) = D_\eta D_\eta V = -[\mathcal{F}(\Psi_\eta), V]_{\Psi_\eta}^\eta. \quad (3.9)$$

With the initial condition  $\mathcal{F}(\Psi_\eta[\tau = 0; V]) = 0$ , this differential equation ensures  $\mathcal{F}(\Psi_\eta[\tau; V]) = 0$  for arbitrary  $\tau$ , and then (3.8) holds.

### Associated fields $\Psi_d = \Psi_d[V]$

The associated fields  $\Psi_d[V]$  are the functionals satisfying

$$d\Psi_\eta = (-)^d D_\eta \Psi_d. \quad (3.10)$$

To derive the differential equation which defines  $\Psi_d[V]$ , let us introduce

$$\mathcal{I}(\tau) \equiv D_\eta(\tau) \Psi_d[\tau; V] - (-)^d d\Psi_\eta[\tau; V] \quad (3.11)$$

and consider its differentiation by  $\tau$ :

$$\partial_\tau \mathcal{I}(\tau) = [V, \mathcal{I}(\tau)]_{\Psi_\eta}^\eta + D_\eta \left( \partial_\tau \Psi_d - dV - [V, \Psi_d]_{\Psi_\eta}^\eta \right). \quad (3.12)$$

We define the associated fields  $\Psi_d[\tau; V]$  by the following differential equation:

$$\partial_\tau \Psi_d[\tau; V] = dV + [V, \Psi_d[\tau; V]]_{\Psi_\eta[\tau; V]}^\eta, \quad (3.13)$$

with the initial condition  $\Psi_d[\tau = 0; V] = 0$ . Then, the equation (3.12) becomes  $\partial_\tau \mathcal{I}(\tau) = [V, \mathcal{I}(\tau)]_{\Psi_\eta}^\eta$ , and leads to the vanishing of  $\mathcal{I}(\tau)$  at arbitrary  $\tau$  since  $\mathcal{I}(\tau = 0) = 0$ , which means that the associated field  $\Psi_d$  defined by (3.13) actually satisfies the WZW-like relation (3.10).

### Action $S_\eta = S_\eta[V]$

Utilizing the pure gauge string field  $\Psi_\eta[V]$  and associated fields  $\Psi_d[V]$  which are defined by (3.5) and (3.13) to satisfy the WZW-like relations (3.8) and (3.10), one can construct a new gauge invariant action as follows:

$$S_\eta[V] = \int_0^1 dt \langle \Psi_t[V(t)], Q \Psi_\eta[V(t)] \rangle. \quad (3.14)$$

The variation of the action can be taken in the same manner as that in section 2.3:

$$\delta S_\eta[V] = \langle \Psi_\delta[V], Q \Psi_\eta[V] \rangle, \quad (3.15)$$

and the equation of motion can be read off from it:

$$Q \Psi_\eta[V] = 0. \quad (3.16)$$

Since  $D_\eta$  and  $Q$  are nilpotent, the action is invariant under the following gauge transformations:

$$\Psi_\delta[V] = D_\eta \Omega + Q \Lambda, \quad (3.17)$$

where  $\Omega$  and  $\Lambda$  are the gauge parameters belonging to the large Hilbert space, which carry ghost numbers 0 and picture numbers 1 and 0 respectively.

## **3.2 Equivalence of the actions in the different parameterizations**

In this subsection, we show the equivalence of the actions  $S_\eta[V]$  and  $S_{\text{EKS}}[\Phi]$ , by identifying the pure-gauge-like string fields  $A_\eta$ . We also derive the relation of two dynamical string fields  $\Phi$  and

$V$  from the identification of the pure-gauge-like fields. Let us consider the identification of the pure-gauge-like fields in the both parameterizations:

$$\pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}) = \Psi_\eta[\Phi(t)] \equiv \Psi_\eta[V(t)] = \int_0^1 d\tau D_\eta(\tau) V(t). \quad (3.18)$$

Apparently it provides the equivalence of the equations of motions  $Q\Psi_\eta[V] = Q\Psi_\eta[\Phi]$ . In addition, one can see that it also provides the equivalence of the two actions in the different parameterization.

### Approach 1

Under  $\Psi_\eta[\Phi] \equiv \Psi_\eta[V]$ , the associated fields in two parameterizations are equivalent up to  $D_\eta$ -exact terms  $\Psi_d[\Phi] = \Psi_d[V] + (D_\eta\text{-exact terms})$ , which is guaranteed by the WZW-like relation  $(-)^d d\Psi_\eta = D_\eta \Psi_d$ :

$$D_\eta(\Psi_d[\Phi(t)] - \Psi_d[V(t)]) = (-)^d d(\Psi_\eta[\Phi(t)] - \Psi_\eta[V(t)]) = 0. \quad (3.19)$$

The  $D_\eta$ -exact terms in the associated fields do not affect to the WZW-like relation  $(-)^d d\Psi_\eta = D_\eta \Psi_d$ . Recall that there exists the arbitrariness to add  $D_\eta$ -exact terms in the associated fields. Besides, since  $Q\Psi_\eta$  is  $D_\eta$ -exact, the difference between  $\Psi_t[\Phi]$  and  $\Psi_t[V]$  does not contribute to the action. Then, two actions are shown to be equivalent:

$$S_{\text{EKS}}[\Phi] = \int_0^1 dt \langle \Psi_t[\Phi(t)], Q\Psi_\eta[\Phi(t)] \rangle = \int_0^1 dt \langle \Psi_t[V(t)], Q\Psi_\eta[V(t)] \rangle = S_\eta[V]. \quad (3.20)$$

The correspondence of the gauge parameters is given by (2.66) and (2.67), with the mixing of the trivial transformation (2.68). Note that we only use the WZW-like relations here, and therefore this identification provides the equivalence of the WZW-like actions in the arbitrary parameterizations as long as the WZW-like relations hold.

### Approach 2

Let us consider a linear operator  $f$  defined by

$$f \equiv \sum_{n=0}^{\infty} (\xi(D_\eta - \eta))^n. \quad (3.21)$$

One can quickly check this  $f$  satisfies  $\llbracket f\xi, D_\eta \rrbracket = 0$ . Since  $Q\Psi_\eta = -D_\eta \Psi_Q$ ,  $(D_\eta)^2 = 0$ , and

$$\begin{aligned} \Psi_t &= (f\xi D_\eta + D_\eta f\xi)\Psi_t = f\xi \partial_t \Psi_\eta + D_\eta f\xi \Psi_t \\ &= \sum_{n=0}^{\infty} (\xi(D_\eta - \eta))^n \xi \partial_t \Psi_\eta + D_\eta f\xi \Psi_t, \end{aligned} \quad (3.22)$$

we can rewrite the action as follows:

$$S_\eta = \int_0^1 dt \langle \Psi_t(t), Q\Psi_\eta(t) \rangle = \sum_{n=0}^{\infty} \int_0^1 dt \langle (\xi(D_\eta(t) - \eta))^n \xi \partial_t \Psi_\eta(t), Q\Psi_\eta(t) \rangle. \quad (3.23)$$

This form of the WZW-like action consists of fields  $\Psi_\eta$  and operators  $Q$ ,  $\xi$ , and  $\partial_t$ . Hence the identification (3.18) automatically provides the equivalence of two actions.

## Relation of the fields $\Phi$ and $V$ , and Partial-gauge-fixing

The identification  $\Psi_\eta[\Phi] \equiv \Psi_\eta[V]$  can be solved by  $\Phi$ . Exponentiating both hand side:  $e^{\wedge\Psi_\eta[\Phi]} = e^{\wedge\Psi_\eta[V]}$ , and using the property of the cohomomorphism and group-like element, the condition becomes

$$e^{\wedge\Psi_\eta[\Phi]} = e^{\wedge\pi_1\widehat{\mathbf{G}}(e^{\wedge\Phi})} = \widehat{\mathbf{G}}e^{\wedge\Phi} = e^{\wedge\Psi_\eta[V]}. \quad (3.24)$$

Since  $\widehat{\mathbf{G}}$  is invertible, by acting  $\widehat{\mathbf{G}}^{-1}$  and projecting by  $\pi_1$ , the condition which provides  $S_\eta[V]$  from the  $L_\infty$ -action  $S_{\text{EKS}}[\Phi]$  is obtained:

$$\Phi[V] = \pi_1\widehat{\mathbf{G}}^{-1}(e^{\wedge\Psi_\eta[V]}). \quad (3.25)$$

Expanding it in powers of  $V$ , it reads

$$\begin{aligned} \Phi[V] = \eta V - \frac{1}{2}\eta\lambda_2^{[0]}(V, \eta V) + \frac{1}{12}\eta\Big( -\lambda_3^{[0]}(V, \eta V, \eta V) + 2\lambda_2^{[0]}(\lambda_2^{[0]}(V, \eta V), \eta V) \\ - \lambda_2^{[0]}(V, \lambda_2^{[0]}(\eta V, \eta V)) + 2\lambda_2^{[0]}(V, \eta\lambda_2^{[0]}(V, \eta V)) \Big) + O(V^4). \end{aligned} \quad (3.26)$$

The identification  $\Psi_\eta[\Phi] \equiv \Psi_\eta[V]$  can be solved also by  $V$  when the  $\eta$ -symmetry is fixed. By expanding  $V = V[\Phi] = V_1(\Phi) + V_2(\Phi, \Phi) + V_3(\Phi, \Phi, \Phi) + \dots$  in powers of  $\Phi$ , and acting  $\xi$  on the both hand sides of  $\Psi_\eta[\Phi] \equiv \Psi_\eta[V]$ , one can determine  $V_n$  perturbatively. The simplest choice of the partial-gauge-fixing condition is  $\xi V = 0$ , which provides  $\xi\eta V = V$ . Then the explicit form of the partially-gauge-fixed string field  $V(\Phi)$  which provides the  $L_\infty$ -action  $S_{\text{EKS}}[\Phi]$  from  $S_\eta[V]$  is obtained as follows:

$$\begin{aligned} V[\Phi] = \xi\Phi + \frac{1}{2}\xi\eta\lambda_2^{[0]}(\xi\Phi, \Phi) + \frac{1}{12}\xi\eta\Big( \lambda_3^{[0]}(\xi\Phi, \Phi, \Phi) - 2\lambda_2^{[0]}(\lambda_2^{[0]}(\xi\Phi, \Phi), \Phi) + \lambda_2^{[0]}(\xi\Phi, \lambda_2^{[0]}(\Phi, \Phi)) \\ + \lambda_2^{[0]}(\xi\Phi, \eta\lambda_2^{[0]}(\xi\Phi, \Phi)) + 3\lambda_2^{[0]}(\xi\eta\lambda_2^{[0]}(\xi\Phi, \Phi), \Phi) \Big) + O(V^4). \end{aligned} \quad (3.27)$$

If we choose the form of  $\mathbf{G}$  as given in [7], it reads

$$\begin{aligned} V[\Phi] = \xi\Phi + \frac{1}{3!}\xi[\xi\Phi, \Phi] + \frac{1}{4!}\Big( \frac{1}{4}\xi[X\xi\Phi, \Phi, \Phi] + \frac{1}{4}\xi X[\xi\Phi, \Phi, \Phi] + \frac{1}{2}\xi[X\Phi, \xi\Phi, \Phi] \\ + \frac{1}{3}\xi[\xi\Phi, \xi[\Phi, \Phi]] - \frac{2}{3}\xi[\xi\Phi, [\xi\Phi, \Phi]] + \frac{4}{3}\xi[\Phi, \xi[\xi\Phi, \Phi]] \Big) + O(V^4). \end{aligned} \quad (3.28)$$

## 4 Conclusions and Discussions

In this paper, we clarified a WZW-like structure naturally arising from the  $L_\infty$  formulation, where the pure-gauge-like field  $\Psi_\eta[\Phi] \equiv \pi_1\widehat{\mathbf{G}}(e^{\wedge\Phi})$  and the associated fields  $\Psi_d[\Phi] \equiv \pi_1\widehat{\mathbf{G}}(\xi_d e^{\wedge\Phi})$  satisfy the  $\mathbb{Z}_2$ -reversed version of the conventional WZW-like relations given in [19]. Once WZW-like functionals  $\Psi_\eta[\varphi]$  and  $\Psi_d[\varphi]$  of some dynamical string field  $\varphi$  are given, we can construct a gauge invariant WZW-like action  $S_\eta[\varphi]$  in terms of them. We showed that the  $L_\infty$ -action  $S_{\text{EKS}}$  just gives the WZW-like action  $S_\eta[\Phi]$  parameterized by  $\Phi$ . We also provided the alternative parameterization of the WZW-like action  $S_\eta[V]$  by the string field in the large Hilbert space  $V$ . The pure-gauge-like field  $\Psi_\eta[V]$  and the associated fields  $\Psi_d[V]$  can be defined by the differential equations which are the  $\mathbb{Z}_2$ -reversed versions of those in [19]. We concluded the equivalence of our new action  $S_\eta[V]$  and the  $L_\infty$  action  $S_\eta[\Phi] \equiv S_{\text{EKS}}$  on the basis of the procedure demonstrated in [29]. The direct relation between two dynamical string fields  $\Phi$  and  $V$ , a field redefinition, can be derived from this equivalence with partially gauge-fixing of  $V$  or trivial uplift of  $\Phi$ .

## Towards the equivalence to the conventional WZW-like action

Although we expect the equivalence between our WZW-like action  $S_\eta[V]$  parametrized by  $V$  and the conventional WZW-like action  $S_{\text{wzw}}[V_c]$  parametrized by  $V_c$ :

$$\begin{aligned} S_\eta[V] &= \frac{1}{2}\langle V, Q\eta V \rangle + \frac{\kappa}{3!}\langle V, Q[V, \eta V]^\eta \rangle + \frac{\kappa^2}{4!}\langle V, Q([V, \eta V, \eta V]^\eta + [V, [V, \eta V]^\eta]^\eta) \rangle + \cdots, \\ S_{\text{wzw}}[V_c] &= \frac{1}{2}\langle \eta V_c, QV_c \rangle + \frac{\kappa}{3!}\langle \eta V_c, [V_c, QV_c] \rangle + \frac{\kappa^2}{4!}\langle \eta V_c, ([V_c, QV_c, QV_c] + [V_c, [V_c, QV_c]]) \rangle + \cdots, \end{aligned}$$

where dynamical heterotic NS string fields  $V$  and  $V_c$  both belong to the large Hilbert space, it remains to be understood. At least perturbatively, one can show it as follows. We identify  $V_c = V + O(\kappa^2)$  because of  $\mathbf{L}_2^\eta = -\mathbf{L}_2^{\text{BOS}}$ , which implies that the first nontrivial order is  $\kappa^2$ , namely the quartic interaction. Let us see how the equivalence can be shown in this order. It would be crucial that the 3-products  $\mathbf{L}_3^\eta$  and  $\mathbf{L}_3^{\text{B}}$  are made from the same gauge product  $\lambda_3^{[1]}$ :

$$\mathbf{L}_3^\eta = -\frac{1}{2}[\![\mathbf{Q}, \lambda_3^{[1]}]\!], \quad \mathbf{L}_3^{\text{BOS}} = \frac{1}{2}[\![\boldsymbol{\eta}, \lambda_3^{[1]}]\!].$$

Utilizing them, the quartic interactions in both actions can be written in term of  $\lambda_3^{[1]}$  as

$$\langle V, Q[V, \eta V, \eta V]^\eta \rangle = -\frac{1}{2}\langle QV, \lambda_3^{[1]}(QV, \eta V, \eta V) \rangle + \langle QV, \lambda_3^{[1]}(V, Q\eta V, \eta V) \rangle, \quad (4.1)$$

$$\langle \eta V_c, [V_c, QV_c, QV_c] \rangle = -\frac{1}{2}\langle \eta V_c, \lambda_3^{[1]}(\eta V_c, QV_c, QV_c) \rangle - \langle \eta V_c, \lambda_3^{[1]}(V_c, Q\eta V_c, QV_c) \rangle. \quad (4.2)$$

The difference between the quartic interactions comes from the second terms of (4.1) and (4.2):

$$S_{\eta,4}[V] - S_{\text{wzw},4}[V_c] = \frac{\kappa^2}{4!}\langle Q\eta V, 2\lambda_3^{[1]}(V, \eta V, QV) \rangle + O(\kappa^3), \quad (4.3)$$

which can be compensated by the following off-shell field redefinition:

$$V_c = V + \frac{2\kappa^2}{4!}\lambda_3^{[1]}(V, \eta V, QV) + O(\kappa^3). \quad (4.4)$$

Thus, under the identification of the string fields  $V_c$  and  $V$  by (4.4), two actions  $S_\eta[V]$  and  $S_{\text{wzw}}[V_c]$  are equivalent at  $\kappa^2$ .

To prove their all-order equivalence, it will be helpful to characterize the conventional WZW-like action in terms of the gauge products  $\lambda_N^{[d]}$ . In fact,  $Q_G$ , the pure-gauge shifted BRST operator which plays a key role in conventional WZW-like formulation, can be written as  $Q_G = \mathcal{E}_V^{-1}Q\mathcal{E}_V$  using a linear map  $\mathcal{E}_V$  [32]. It would provide a solution of this problem to understand the relation between two cohomomorphisms  $\mathcal{E}_V$  and  $\widehat{\mathbf{G}}$ .

## Ramond sector of closed superstring fields

The concept of the WZW-like structure based on the dual product  $\mathbf{L}^\eta$  will play crucial roles in construction of the complete action of heterotic string field theory including both NS and R sectors. As [2], our WZW-like action  $S_\eta[V]$  will provide a good starting point of the construction of complete actions. It is expected that as demonstrated in [4] for open superstrings, the WZW-like structure including the Ramond sector would be written by the  $L_\infty$  products  $\mathbf{L}^\eta = \widehat{\mathbf{G}}\boldsymbol{\eta}\widehat{\mathbf{G}}^{-1}$  which are dual to the  $L_\infty$  products of [8]:  $\widetilde{\mathbf{L}} = \widehat{\mathbf{G}}^{-1}(\mathbf{Q} + \dots)\widehat{\mathbf{G}}$  satisfying  $[\![\boldsymbol{\eta}, \widetilde{\mathbf{L}}]\!] = 0$ .

It is also expected that the action for closed NS-R and R-NS strings can be constructed in the same manner as that for heterotic string. For closed R-R strings, it is not clear whether or how the kinetic term can be constructed with no constraint yet. If the kinetic term can be constructed, it may be possible to construct the complete action of type II string on the basis of the concept of the WZW-like structure arising from the dual product  $\mathbf{L}^\eta$ .

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# Appendices

## A Basic facts of $A_\infty$ and $L_\infty$

In this section, we give a short review of coalgebraic description for  $A_\infty/L_\infty$  algebras. For more details, see [40–42] or some mathematical manuscripts.

### Coalgebras: tensor algebra $\mathcal{T}(\mathcal{H})$ and its symmetrization $\mathcal{S}(\mathcal{H})$

Let  $\mathcal{C}$  be a set. Suppose that a coproduct  $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes' \mathcal{C}$  is defined on  $\mathcal{C}$  and it is coassociative

$$(\Delta \otimes' \mathbb{1})\Delta = (\mathbb{1} \otimes' \Delta)\Delta. \quad (\text{A.1})$$

Then, the pair  $(\mathcal{C}, \Delta)$  is called a *coalgebra*. We write  $\mathcal{T}(\mathcal{H})$  for a *tensor algebra* of a graded vector space  $\mathcal{H}$ :

$$\mathcal{T}(\mathcal{H}) = \mathcal{H}^{\otimes 0} \oplus \mathcal{H}^{\otimes 1} \oplus \mathcal{H}^{\otimes 2} \oplus \dots \quad (\text{A.2})$$

One can define a coassociative coproduct  $\Delta : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H}) \otimes' \mathcal{T}(\mathcal{H})$  by

$$\Delta(\Phi_1 \otimes \dots \otimes \Phi_n) \equiv \sum_{k=0}^n (\Phi_1 \otimes \dots \otimes \Phi_k) \otimes' (\Phi_{k+1} \otimes \dots \otimes \Phi_n) \quad (\text{A.3})$$

with  $\otimes' \equiv \otimes$ , where  $\Phi_1 \otimes \dots \otimes \Phi_n \in \mathcal{H}^{\otimes n} \subset \mathcal{T}(\mathcal{H})$ . Then, the pair  $(\mathcal{T}(\mathcal{H}), \Delta)$  gives a coalgebra. As well as  $\mathcal{T}(\mathcal{H})$ , its symmetrization  $\mathcal{S}(\mathcal{H})$  also gives a coalgebra. Recall that the *symmetrized tensor product*  $\wedge$  for  $\Phi_1, \Phi_2 \in \mathcal{H}$  is defined by

$$\Phi_1 \wedge \Phi_2 \equiv \Phi_1 \otimes \Phi_2 - (-1)^{\deg(\Phi_1)\deg(\Phi_2)} \Phi_2 \otimes \Phi_1, \quad (\text{A.4})$$

and it satisfies the following properties for  $\Phi_1, \dots, \Phi_n \in \mathcal{H}$ :

$$\Phi_1 \wedge \Phi_2 = (-1)^{\deg(\Phi_1)\deg(\Phi_2)} \Phi_2 \wedge \Phi_1, \quad (\text{A.5})$$

$$(\Phi_1 \wedge \Phi_2) \wedge \Phi_3 = \Phi_1 \wedge (\Phi_2 \wedge \Phi_3), \quad (\text{A.6})$$

$$\Phi_1 \wedge \Phi_2 \wedge \dots \wedge \Phi_n = \sum_{\sigma} (-1)^\sigma \Phi_{\sigma(1)} \otimes \Phi_{\sigma(2)} \otimes \dots \otimes \Phi_{\sigma(n)}. \quad (\text{A.7})$$

We write  $\mathcal{H}^{\wedge n}$  for the vector space spanned by  $n$ -fold symmetrized tensor  $\Phi_1 \wedge \cdots \wedge \Phi_n$ , and  $\mathcal{S}(\mathcal{H})$  for the symmetrization of  $\mathcal{T}(\mathcal{H})$ , which is called the *symmetrized tensor algebra*  $\mathcal{S}(\mathcal{H})$ :

$$\mathcal{S}(\mathcal{H}) = \mathcal{H}^{\wedge 0} \oplus \mathcal{H}^{\wedge 1} \oplus \mathcal{H}^{\wedge 2} \oplus \cdots. \quad (\text{A.8})$$

A coassociative coproduct  $\Delta : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H}) \otimes' \mathcal{S}(\mathcal{H})$  can be defined by

$$\Delta(\Phi_1 \wedge \cdots \wedge \Phi_n) \equiv \sum_{k=0}^n \sum_{\sigma}' (-)^{\sigma} (\Phi_{\sigma(1)} \wedge \cdots \wedge \Phi_{\sigma(k)}) \otimes' (\Phi_{\sigma(k+1)} \wedge \cdots \wedge \Phi_{\sigma(n)}). \quad (\text{A.9})$$

with  $\otimes' \equiv \otimes$ , where  $\Phi_1 \wedge \cdots \wedge \Phi_n \in \mathcal{H}^{\wedge n}$  and  $\sigma$  runs over  $(k, n-k)$ -unshuffle. Then, the pair  $(\mathcal{S}(\mathcal{H}), \Delta)$  gives coalgebra.

In the case of open superstring field theory,  $\mathcal{H}$  is the state space of open superstrings, which is a  $\mathbb{Z}_2$ -graded vector space, and its grading which we call degree is given by the Grassmann parity minus one mod 2. Then, the pair  $(\mathcal{T}(\mathcal{H}), \Delta)$  is the Fock space of open superstrings. On the other hand, in the case of closed superstring field theory,  $\mathcal{H}$  is the state space of closed superstrings, which is a  $\mathbb{Z}_2$ -graded vector space, and its grading which we call degree is given by the Grassmann parity mod 2. Then, the pair  $(\mathcal{S}(\mathcal{H}), \Delta)$  is the Fock space of closed superstrings.

#### Multi-linear maps as a coderivation

A linear operator  $m : \mathcal{C} \rightarrow \mathcal{C}$  which raise the degree one is called *coderivation* if it satisfies

$$\Delta m = (m \otimes \mathbb{1})\Delta + (\mathbb{1} \otimes m)\Delta. \quad (\text{A.10})$$

Multilinear maps with degree 1 and 0 naturally induce the maps from  $\mathcal{T}(\mathcal{H})$  to  $\mathcal{T}(\mathcal{H})$  or from  $\mathcal{S}(\mathcal{H})$  to  $\mathcal{S}(\mathcal{H})$ . They are called a coderivation and a cohomomorphism respectively.

Recall that from a  $n$ -fold multilinear product  $b_n$  of  $\Phi_1, \dots, \Phi_n \in \mathcal{H}$ , one can define a linear map on  $\mathcal{H}^{\otimes n}$ , which we write  $b_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}$ , by

$$b_n(\Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_n) \equiv b_n(\Phi_1, \Phi_2, \dots, \Phi_n), \quad (\text{A.11})$$

where  $\Phi_1 \otimes \cdots \otimes \Phi_n \in \mathcal{H}^{\otimes n}$ . Let  $A : \mathcal{H}^{\otimes k} \rightarrow \mathcal{H}^{\otimes l}$  and  $B : \mathcal{H}^{\otimes m} \rightarrow \mathcal{H}^{\otimes n}$  be multilinear maps. One can naturally define a product of these maps  $A \otimes B : \mathcal{H}^{\otimes k+m} \rightarrow \mathcal{H}^{\otimes l+n}$  by

$$A \otimes B(\Phi_1 \otimes \cdots \otimes \Phi_{k+m}) = \sum_k (-)^{B(\Phi_1 + \cdots + \Phi_k)} A(\Phi_1 \otimes \cdots \otimes \Phi_k) \otimes B(\Phi_{k+1} \otimes \cdots \otimes \Phi_{k+m}). \quad (\text{A.12})$$

Then, the identity operator on  $\mathcal{H}^{\otimes n}$  and  $\mathcal{H}^{\wedge n}$  is defined by

$$\mathbb{I}_n = \overbrace{\mathbb{I} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}}^n = \frac{1}{n!} \overbrace{\mathbb{I} \wedge \mathbb{I} \wedge \cdots \wedge \mathbb{I}}^n. \quad (\text{A.13})$$

Using a degree one map  $b_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}$ , one can naturally construct a coderivation  $\mathbf{b}_n : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$  by

$$\mathbf{b}_n \Phi = \sum_{k=0}^{N-n} (\mathbb{I}_k \otimes b_n \otimes \mathbb{I}_{N-n-k}) \Phi, \quad \Phi \in \mathcal{H}^{\otimes N \geq n} \subset \mathcal{T}(\mathcal{H}), \quad (\text{A.14})$$



and  $\mathbf{b}_n$  vanishes when acting on  $\mathcal{H}^{\otimes N \leq n}$ .

Similarly, a symmetric multilinear product  $b_n$  of  $\Phi_1, \dots, \Phi_n \in \mathcal{H}$  naturally defines a map  $b_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}$  by

$$b_n(\Phi_1 \wedge \Phi_2 \wedge \dots \wedge \Phi_n) = b_n(\Phi_1, \Phi_2, \dots, \Phi_n). \quad (\text{A.15})$$

The symmetric tensor product of two multilinear maps  $A : \mathcal{H}^{\wedge k} \rightarrow \mathcal{H}^{\wedge l}$  and  $B : \mathcal{H}^{\wedge m} \rightarrow \mathcal{H}^{\wedge n}$ ,  $A \wedge B : \mathcal{H}^{\wedge k+m} \rightarrow \mathcal{H}^{\wedge l+n}$ , can also be defined naturally by

$$A \wedge B(\Phi_1 \wedge \dots \wedge \Phi_{k+m}) = \sum_{\sigma}' (-)^{\sigma} A(\Phi_{\sigma(1)} \wedge \dots \wedge \Phi_{\sigma(k)}) \wedge B(\Phi_{\sigma(k+1)} \wedge \dots \wedge \Phi_{\sigma(k+m)}). \quad (\text{A.16})$$

A degree map  $b_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}$  naturally gives a coderivation  $\mathbf{b}_n : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H})$  defined by

$$\mathbf{b}_n \Phi = (b_n \wedge \mathbb{I}_{N-n}) \Phi, \quad \Phi \in \mathcal{H}^{\wedge N \geq n} \subset \mathcal{S}(\mathcal{H}), \quad (\text{A.17})$$

and  $\mathbf{b}_n$  vanishes when acting on  $\mathcal{H}^{\wedge N \leq n}$ . For example, we find that  $\mathbf{b}_1$  acts as follows:

$$\begin{aligned} \mathbf{b}_1 : \quad 1 &\rightarrow 0 \\ \Phi_1 &\rightarrow b_1(\Phi_1) \\ \Phi_1 \wedge \Phi_2 &\rightarrow b_1(\Phi_1) \wedge \Phi_2 + (-)^{\deg(\Phi_1)\deg(b_1)} \Phi_1 \wedge b_1(\Phi_2). \end{aligned} \quad (\text{A.18})$$

In particular, the coderivation  $\mathbf{b}_0 : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H})$  derived from a map  $b_0 : \mathcal{H}^0 \rightarrow \mathcal{H}$  is given by

$$\mathbf{b}_0 \Phi = (b_0 \wedge \mathbb{I}_N) \Phi = b_0 \wedge \Phi, \quad \Phi \in \mathcal{H}^{\wedge N} \subset \mathcal{S}(\mathcal{H}) \quad (\text{A.19})$$

and acts as

$$\begin{aligned} \mathbf{b}_0 : \quad 1 &\rightarrow b_0 \\ \Phi_1 &\rightarrow b_0 \wedge \Phi_1 \\ \Phi_1 \wedge \Phi_2 &\rightarrow b_0 \wedge \Phi_1 \wedge \Phi_2. \end{aligned} \quad (\text{A.20})$$

Given two coderivations  $\mathbf{b}_n$  and  $\mathbf{c}_m$  which are derived from  $b_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}$  and  $c_m : \mathcal{H}^{\wedge m} \rightarrow \mathcal{H}$  respectively, the graded commutator  $[\mathbf{b}_n, \mathbf{c}_m]$  becomes the coderivation derived from the map  $[b_n, c_m] : \mathcal{H}^{\wedge n+m-1} \rightarrow \mathcal{H}$  which is defined by

$$[b_n, c_m] = b_n(c_m \wedge \mathbb{I}_{n-1}) - (-)^{\deg(b_n)\deg(c_m)} c_m(b_n \wedge \mathbb{I}_{m-1}). \quad (\text{A.21})$$

### Multilinear maps as a cohomomorphism

Given two coalgebras  $C, C'$ , a *cohomomorphism*  $\widehat{\mathbf{f}} : C \rightarrow C'$  is a map of degree zero satisfying

$$\Delta \widehat{\mathbf{f}} = (\widehat{\mathbf{f}} \otimes' \widehat{\mathbf{f}}) \Delta. \quad (\text{A.22})$$

A set of degree zero multilinear maps  $\{\mathbf{f}_n : \mathcal{H}^{\wedge n} \rightarrow \mathcal{H}'\}_{n=0}^{\infty}$  naturally induces a cohomomorphism  $\widehat{\mathbf{f}} : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H}')$ , which we denote as  $\widehat{\mathbf{f}} = \{\mathbf{f}_n\}_{n=0}^{\infty}$ . Its action on  $\Phi_1 \wedge \dots \wedge \Phi_n \in \mathcal{H}^{\wedge n} \subset \mathcal{S}(\mathcal{H})$  is defined by

$$\begin{aligned} \widehat{\mathbf{f}}(\Phi_1 \wedge \dots \wedge \Phi_n) = \sum_{i \leq n} \sum_{k_1 < \dots < k_i} e^{\wedge \mathbf{f}_0} \wedge \mathbf{f}_{k_1}(\Phi_1, \dots, \Phi_{k_1}) \wedge \mathbf{f}_{k_2-k_1}(\Phi_{k_1+1}, \dots, \Phi_{k_2}) \wedge \\ \dots \wedge \mathbf{f}_{k_i-k_{i-1}}(\Phi_{k_{i-1}+1}, \dots, \Phi_n). \end{aligned} \quad (\text{A.23})$$

Its explicit actions are given as follows:

$$\begin{aligned}\widehat{\mathbf{f}} : \quad 1 &\rightarrow e^{\wedge f_0} \\ \Phi &\rightarrow e^{\wedge f_0} \wedge \mathbf{f}_1(\Phi) \\ \Phi_1 \wedge \Phi_2 &\rightarrow e^{\wedge f_0} \wedge \mathbf{f}_1(\Phi_1) \wedge \mathbf{f}_1(\Phi_2) + e^{\wedge f_0} \wedge \mathbf{f}_2(\Phi_1 \wedge \Phi_2).\end{aligned}\tag{A.24}$$

### Cyclic $A_\infty$ and cyclic $L_\infty$

Let  $\mathcal{H}$  be a graded vector space and  $\mathcal{T}(\mathcal{H})$  be its tensor algebra. A *weak  $A_\infty$ -algebra*  $(\mathcal{H}, \mathbf{M})$  is a coalgebra  $\mathcal{T}(\mathcal{H})$  with a coderivation  $\mathbf{M} = \mathbf{M}_0 + \mathbf{M}_1 + \mathbf{M}_2 + \dots$  satisfying

$$(\mathbf{M})^2 = 0.\tag{A.25}$$

We denote the collection of the multilinear maps  $\{M_k\}_{k \geq 0}$  also by  $\mathbf{M}$ . In particular, if  $\mathbf{M}_0 = 0$ ,  $(\mathcal{H}, \mathbf{M})$  is called an  *$M_\infty$ -algebra*. Note that for fixed  $n$ , the equation (A.25) gives

$$\mathbf{M}_n \cdot \mathbf{M}_1 + \mathbf{M}_{n-1} \cdot \mathbf{M}_2 + \dots + \mathbf{M}_2 \cdot \mathbf{M}_{n-1} + \mathbf{M}_1 \cdot \mathbf{M}_n = 0.\tag{A.26}$$

We can act it on  $B_1 \otimes B_2 \otimes \dots \otimes B_n \in \mathcal{H}^{\otimes n}$  to get the  $A_\infty$  relations for the maps  $\{M_k\}$ :

$$\sum_{k=0}^{n-i} (-)^{B_1 + \dots + B_k} M_{n-i+1}(B_1, \dots, B_k, M_i(B_{k+1}, \dots, B_{k+i}), B_{k+i+1}, \dots, B_n) = 0.\tag{A.27}$$

Let  $\langle \cdot, \cdot \rangle : \mathcal{H}^{\otimes 2} \rightarrow \mathbb{C}$  be the BPZ inner product, which gives the graded symplectic form  $\langle \omega | : \mathcal{H}^{\otimes 2} \rightarrow \mathbb{C}$ :

$$\langle A, B \rangle = (-)^A \langle \omega | A \otimes B.\tag{A.28}$$

Given the operator  $\mathcal{O}_n$ , we can define its BPZ-conjugation  $\mathcal{O}_n^\dagger$  as follows:

$$\langle \omega | \mathbb{I} \otimes \mathcal{O}_n = \langle \omega | \mathcal{O}_n^\dagger \otimes \mathbb{I}.\tag{A.29}$$

A pair  $(\mathcal{T}(\mathcal{H}), \mathbf{M}, \omega)$  is called *cyclic  $A_\infty$ -algebra* if each  $M_n$  is BPZ-odd:

$$M_n^\dagger = -M_n.\tag{A.30}$$

Let  $\mathcal{H}$  be a graded vector space and  $\mathcal{S}(\mathcal{H})$  be its symmetrized tensor algebra. A *weak  $L_\infty$ -algebra*  $(\mathcal{H}, \mathbf{L})$  is a coalgebra  $\mathcal{S}(\mathcal{H})$  with a coderivation  $\mathbf{L} = \mathbf{L}_0 + \mathbf{L}_1 + \mathbf{L}_2 + \dots$  satisfying

$$(\mathbf{L})^2 = 0.\tag{A.31}$$

We denote the collection of the multilinear maps  $\{L_k\}_{k \geq 0}$  also by  $\mathbf{L}$ . In particular, if  $\mathbf{L}_0 = 0$ ,  $(\mathcal{H}, \mathbf{L})$  is called an  *$L_\infty$ -algebra*.

In the case of an  $L_\infty$ -algebra, the part of (A.31) that correspond to an  $n$ -fold multilinear map  $\mathcal{H}^{\wedge n} \rightarrow \mathcal{H}$  is given by

$$\mathbf{L}_n \cdot \mathbf{L}_1 + \mathbf{L}_{n-1} \cdot \mathbf{L}_2 + \dots + \mathbf{L}_2 \cdot \mathbf{L}_{n-1} + \mathbf{L}_1 \cdot \mathbf{L}_n = 0.\tag{A.32}$$

We can act it on  $B_1 \wedge B_2 \wedge \dots \wedge B_n \in \mathcal{H}^{\wedge n}$  to get the  $L_\infty$  relations for the multilinear maps  $\{L_k\}$ :

$$0 = \sum_{i+j=n+1} \sum_{\sigma}' (-)^{\sigma} L_j(L_i(B_{\sigma(1)}, \dots, B_{\sigma(i)}), B_{\sigma(i+1)}, \dots, B_{\sigma(n)}).\tag{A.33}$$

A set  $(\mathcal{S}(\mathcal{H}), \mathbf{L}, \omega)$  is called *cyclic  $L_\infty$ -algebra* if each  $L_n$  is BPZ-odd:

$$L_n^\dagger = -L_n.\tag{A.34}$$

## Projector and group-like element

We can naturally define a *projector*  $\pi_1 : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{H}$  whose action on  $\Phi \in \mathcal{S}(\mathcal{H})$  is given by

$$\pi_1 \Phi = \Phi_1, \Phi = \sum_{n=1}^{\infty} \Phi_1 \wedge \cdots \wedge \Phi_n \in \mathcal{S}(\mathcal{H}). \quad (\text{A.35})$$

Note that  $\pi_1$  acts trivially on  $\mathcal{H}$  and commutes with one-coderivations.

Let  $\mathcal{H}_0$  be the degree zero part of  $\mathcal{H}$ . The following exponential map of  $\Phi \in \mathcal{H}_0$

$$e^{\wedge \Phi} = \mathbf{1} + \Phi + \frac{1}{2} \Phi \wedge \Phi + \frac{1}{3!} \Phi \wedge \Phi \wedge \Phi + \cdots. \quad (\text{A.36})$$

is called a *group-like element*. It satisfies

$$\Delta e^{\wedge \Phi} = e^{\wedge \Phi} \wedge e^{\wedge \Phi}, \quad (\text{A.37})$$

The action of a coderivation on a group-like element is given by

$$\mathbf{b}_n(e^{\wedge \Phi}) = \frac{1}{n!} b_n(\Phi^{\wedge n}) \wedge e^{\wedge \Phi}. \quad (\text{A.38})$$

Where we promise  $0! = 1$ . Note that we can not distinguish a one-coderivation  $\mathbf{b}_1$  derived from a linear map  $b_1 : \mathcal{H} \rightarrow \mathcal{H}, \Phi \mapsto b_1(\Phi)$  and a zero-coderivation  $\mathbf{b}_0$  derived from  $b_0 : \mathcal{H}^0 \rightarrow \mathcal{H}, 1 \mapsto b_0 = b_1(\Phi)$  when acting on group-like element:

$$\mathbf{b}_0(e^{\wedge \Phi}) = b_0 \wedge (e^{\wedge \Phi}) = b_1(\Phi) \wedge (e^{\wedge \Phi}) = \mathbf{b}_1(e^{\wedge \Phi}). \quad (\text{A.39})$$

One of the important property of a cohomomorphism is its action on the group-like element:

$$\widehat{\mathbf{f}}(e^{\wedge \Phi}) = (\widehat{\mathbf{f}} \otimes \widehat{\mathbf{f}}) \Delta e^{\wedge \Phi} = (\widehat{\mathbf{f}} \otimes \widehat{\mathbf{f}}) e^{\wedge \Phi} \wedge e^{\wedge \Phi} = \widehat{\mathbf{f}}(e^{\wedge \Phi}) \wedge \widehat{\mathbf{f}}(e^{\wedge \Phi}). \quad (\text{A.40})$$

We can see that the cohomomorphisms preserves the group-like element :  $\widehat{\mathbf{f}}(e^{\wedge \Phi}) = e^{\wedge \Phi'}$ .

Utilizing the projector and the group-like element, the *Maurer-Cartan element* for an  $L_\infty$ -algebra  $(\mathcal{H}, \mathbf{L})$  is given by

$$\mathcal{F}_\Phi := \pi_1 \mathbf{L}(e^{\wedge \Phi}) = \mathbf{L}_1(\Phi) + \frac{1}{2} \mathbf{L}_2(\Phi \wedge \Phi) + \frac{1}{3!} \mathbf{L}_3(\Phi \wedge \Phi \wedge \Phi) + \cdots. \quad (\text{A.41})$$

The *Maurer-Cartan equation* for an  $L_\infty$ -algebra  $(\mathcal{H}, \mathbf{L})$  is given by  $\mathcal{F}_\Phi = 0$ , which correspond to the on-shell condition in string field theory based on the  $L_\infty$ -algebra  $(\mathcal{H}, \mathbf{L})$ .

## B Derivation of (2.71)

For cyclic cohomomorphism  $\widehat{H}$ , the following relation holds:

$$\langle \omega | \pi_2 = \langle \omega | \pi_2 \widehat{H} = \langle \omega | \nabla(\pi_1 \otimes' \pi_1) \Delta \widehat{H} = \langle \omega | \nabla(\pi_1 \widehat{H} \otimes' \pi_1 \widehat{H}) \Delta, \quad (\text{B.1})$$

where  $\pi_2$  is the projector  $\mathcal{S}(\mathcal{H}) \rightarrow \mathcal{H} \wedge \mathcal{H}$ , and  $\nabla$  is the product of the tensor algebra  $(\mathcal{T}(\mathcal{H}), \nabla)$ .

Let us consider its action

$$\begin{aligned}
& \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \\
& + (-)^{AB} \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \\
& + (-)^{BC} \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi}.
\end{aligned} \tag{B.2}$$

The left hand side vanishes since  $\pi_2$  is acting on  $\mathcal{H}^{\wedge n \geq 3}$ . For the right hand side, the first term of (B.2) becomes

$$\begin{aligned}
& \langle \omega | \nabla(\pi_1 \hat{H} \otimes' \pi_1 \hat{H}) \Delta \left( \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \right) \\
& = \langle \omega | \nabla(\pi_1 \hat{H} \otimes' \pi_1 \hat{H}) \left( \frac{1}{1-\Phi} \otimes' \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \right. \\
& \quad + \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes' \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \\
& \quad + \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi} \otimes' \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \\
& \quad \left. + \frac{1}{1-\Phi} \otimes A \otimes \frac{1}{1-\Phi} \otimes B \otimes \frac{1}{1-\Phi} \otimes C \otimes \frac{1}{1-\Phi} \otimes' \frac{1}{1-\Phi} \right).
\end{aligned} \tag{B.3}$$

Hereafter we assume  $A + B + C = \text{even}$ . The inner product in the large Hilbert space is

$$\langle a, b \rangle = (-)^a \langle \omega | a \otimes b, \tag{B.4}$$

and the 1st term and 4th term of (B.3) cancel since  $\langle a, b \rangle = (-)^{(a+1)(b+1)} \langle b, a \rangle$ . Now, gathering the contribution from all terms of (B.2), and utilizing the identities

$$a \wedge e^{\wedge \Phi} = \frac{1}{1-\Phi} \otimes a \otimes \frac{1}{1-\Phi}, \tag{B.5}$$

$$a \wedge b \wedge e^{\wedge \Phi} = \frac{1}{1-\Phi} \otimes a \otimes \frac{1}{1-\Phi} \otimes b \otimes \frac{1}{1-\Phi} + (-)^{ab} \frac{1}{1-\Phi} \otimes b \otimes \frac{1}{1-\Phi} \otimes a \otimes \frac{1}{1-\Phi}, \tag{B.6}$$

one can obtain the BPZ property for  $\hat{H}$ :

$$0 = (-)^B \langle \pi_1 \hat{H}(A \wedge B \wedge e^{\wedge \Phi}), \pi_1 \hat{H}(C \wedge e^{\wedge \Phi}) \rangle + \langle \pi_1 \hat{H}(A \wedge e^{\wedge \Phi}), \pi_1 \hat{H}(B \wedge C \wedge e^{\wedge \Phi}) \rangle. \tag{B.7}$$

Note that in the computation some terms cancel by the symmetric property of the inner product.

## C Embedding to the large Hilbert space of NS closed string

In this appendix, we consider the trivial embedding of the NS closed string field  $\Phi$  in the  $L_\infty$ -formulation into the string field  $\tilde{V}$  in the large Hilbert space, and see that the trivially embedded action is also WZW-like form. In other words, we provide a parameterization of the pure-gauge-like field and the associated fields by the large space string field  $\tilde{V}$ . In particular, we discuss the relation between the gauge transformations in two representations.

We define the trivial embedding by replacing  $\xi\Phi$  to  $\tilde{V}$  which is a string field belonging to the large Hilbert space, so that under the partial-gauge-fixing condition  $\xi\tilde{V} = 0$ , the trivial solution

$\tilde{V} = \xi\Phi$  reproduce the  $L_\infty$ -formulation parameterized by  $\Psi$ , the string field in the small Hilbert space. The action for  $\tilde{V}$  is given by

$$\begin{aligned} S_{\text{EKS}}[\tilde{V}] &= \int_0^1 dt \langle \partial_t \tilde{V}(t), \pi_1(\mathbf{L}(e^{\wedge\eta\tilde{V}(t)})) \rangle \\ &= \int_0^1 dt \langle \pi_1 \hat{\mathbf{G}}((\partial_t \tilde{V}(t)) \wedge e^{\wedge\eta\tilde{V}(t)}), Q \pi_1 \hat{\mathbf{G}}(e^{\wedge\eta\tilde{V}(t)}) \rangle. \end{aligned} \quad (\text{C.1})$$

One can confirm the functionals appearing in the action

$$\Psi_\eta[\tilde{V}] = \pi_1 \hat{\mathbf{G}}(e^{\wedge\eta\tilde{V}}), \quad (\text{C.2})$$

$$\Psi_d[\tilde{V}] = \pi_1 \hat{\mathbf{G}}(d\tilde{V} \wedge e^{\wedge\eta\tilde{V}}) \quad (\text{for } d = \partial_t, \delta) \quad (\text{C.3})$$

satisfy the WZW-like relations:

$$0 = \pi_1 \mathbf{L}^\eta(e^{\wedge\Psi_\eta[\tilde{V}]}) \quad (-)^d d \Psi_\eta[\tilde{V}] = D_\eta \Psi_d[\tilde{V}]. \quad (\text{C.4})$$

The first relation follows from the projection invariance  $e^{\wedge\eta\tilde{V}} = \eta\xi e^{\wedge\eta\tilde{V}}$  as (2.49). To check the second relation, consider

$$\Psi_d[\tilde{V}] = \pi_1 \hat{\mathbf{G}}(d(\eta\xi + \xi\eta)\tilde{V} \wedge e^{\wedge\eta\tilde{V}}) = \pi_1 \hat{\mathbf{G}}(d\xi\eta\tilde{V} \wedge e^{\wedge\eta\tilde{V}}) + (-)^d D_\eta \pi_1 \hat{\mathbf{G}}(d\xi\tilde{V} \wedge e^{\wedge\eta\tilde{V}}), \quad (\text{C.5})$$

where we used

$$\begin{aligned} (-)^d \pi_1 \hat{\mathbf{G}}(d\xi\eta\tilde{V} \wedge e^{\wedge\eta\tilde{V}}) &= \pi_1 \hat{\mathbf{G}}_\eta(d\xi\tilde{V} \wedge e^{\wedge\eta\tilde{V}}) = \pi_1 \mathbf{L}^\eta \hat{\mathbf{G}}(d\xi\tilde{V} \wedge e^{\wedge\eta\tilde{V}}) \\ &= \pi_1 \mathbf{L}^\eta (\pi_1 \hat{\mathbf{G}}(d\xi\tilde{V} \wedge e^{\wedge\eta\tilde{V}}) \wedge \hat{\mathbf{G}}e^{\wedge\eta\tilde{V}}) = D_\eta \pi_1 \hat{\mathbf{G}}(d\xi\tilde{V} \wedge e^{\wedge\eta\tilde{V}}). \end{aligned} \quad (\text{C.6})$$

Since the second term vanishes when acted by  $D_\eta$ , the same computation as (2.51) can be applied to show  $(-)^d d \Psi_\eta[\tilde{V}] = D_\eta \Psi_d[\tilde{V}]$ .

In fact, the embedding procedure  $\xi\Phi \rightarrow \tilde{V}$  has an ambiguity: one can consider the replacing  $\xi\Phi = \xi\eta\xi\Phi \rightarrow \xi\eta\tilde{V}$ . However, the difference of  $\Psi_d$  produced by the difference of the procedure is only  $D_\eta$ -exact term as seen in (C.6), which does not affect to the WZW-like relation  $(-)^d d \Psi_\eta = D_\eta \Psi_d$ . Besides, since  $Q\Psi_\eta$  is  $D_\eta$ -exact, the  $D_\eta$ -exact part of  $\Psi_t$  does not affect the action, that is, the action does not depend on this ambiguity. Here we fix this ambiguity by requiring the invertibility of  $\Psi_{d=\delta, \partial_t}$  as functions of  $d\tilde{V}$  [31].

For derivation  $d$  which does not commute with  $\hat{\mathbf{G}}$ , such as  $Q$ , we define  $\Psi_d[\tilde{V}]$  by

$$\Psi_d[\tilde{V}] = \pi_1 \hat{\mathbf{G}}(\pi_1 \hat{\mathbf{G}}^{-1} d \hat{\mathbf{G}}(\tilde{V} \wedge \int e^{\wedge\eta\tilde{V}}) \wedge e^{\wedge\eta\tilde{V}}), \quad (\text{C.7})$$

where

$$\int e^{\wedge\eta\tilde{V}} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\eta\tilde{V})^{\wedge n}. \quad (\text{C.8})$$

For  $d = Q$ , it reads

$$\Psi_Q[\tilde{V}] = \pi_1 \hat{\mathbf{G}}(\pi_1 \mathbf{L}(\tilde{V} \wedge \int e^{\wedge\eta\tilde{V}}) \wedge e^{\wedge\eta\tilde{V}}). \quad (\text{C.9})$$

To check the WZW-like relation, utilizing  $\widehat{\mathbf{G}}^{-1}d\widehat{\mathbf{G}} = (-)^d\llbracket\boldsymbol{\eta}, \boldsymbol{\xi}_d\rrbracket$  we write  $\Psi_d[\widetilde{V}]$  as

$$\begin{aligned}\Psi_d[\widetilde{V}] &= \pi_1\widehat{\mathbf{G}}\left(\pi_1\widehat{\mathbf{G}}^{-1}d\widehat{\mathbf{G}}(\widetilde{V} \wedge \int e^{\wedge\eta\widetilde{V}}) \wedge e^{\wedge\eta\widetilde{V}}\right) \\ &= \pi_1\widehat{\mathbf{G}}\left(\pi_1(-)^d\llbracket\boldsymbol{\eta}, \boldsymbol{\xi}_d\rrbracket(\widetilde{V} \wedge \int e^{\wedge\eta\widetilde{V}}) \wedge e^{\wedge\eta\widetilde{V}}\right) \\ &= \pi_1\widehat{\mathbf{G}}\left(\pi_1\boldsymbol{\xi}_d e^{\wedge\eta\widetilde{V}} \wedge e^{\wedge\eta\widetilde{V}}\right) + (-)^d D_\eta \pi_1\widehat{\mathbf{G}}\left(\pi_1\boldsymbol{\xi}_d(\widetilde{V} \wedge \int e^{\wedge\eta\widetilde{V}}) \wedge e^{\wedge\eta\widetilde{V}}\right).\end{aligned}\quad (\text{C.10})$$

Since the second term vanishes when acted by  $D_\eta$ , the same computation as (2.51) can be applied to show  $(-)^d d\Psi_\eta[\widetilde{V}] = D_\eta \Psi_d[\widetilde{V}]$ .

Thus, the action is written in the WZW-like form:

$$S_{\text{EKS}}[\widetilde{V}] = \int_0^1 dt \langle \Psi_t[\widetilde{V}(t)], Q\Psi_\eta[\widetilde{V}(t)] \rangle. \quad (\text{C.11})$$

Its variation and the on-shell condition can be taken by parallel computations in section 2.3.

### Gauge transformations

The variation of the action  $S_{\text{EKS}}[\widetilde{V}]$  in the form of (C.1) can be taken as

$$\delta S_{\text{EKS}}[\widetilde{V}] = \langle \delta\widetilde{V}, \pi_1(\mathbf{L}(e^{\wedge\eta\widetilde{V}})) \rangle, \quad (\text{C.12})$$

and one can find the action  $S_{\text{EKS}}[\widetilde{V}]$  is invariant under the gauge transformations<sup>8</sup>

$$\delta\widetilde{V} = -\eta\xi\omega - \pi_1\mathbf{L}(\xi\lambda \wedge e^{\wedge\eta\widetilde{V}}), \quad (\text{C.13})$$

where  $\omega$  and  $\lambda$  are the gauge parameters belonging to the small Hilbert space, which carry ghost numbers 1 and 1, and picture numbers 0 and  $-1$ , respectively. The minus sign is a convention in which  $\eta\delta\widetilde{V}$  becomes  $\delta\Phi$  of (2.24). In the following, we will discuss how this gauge symmetry can be written in the WZW-like form:

$$\Psi_\delta[\widetilde{V}] = D_\eta\Omega[\omega, \widetilde{V}] + Q\Lambda[\lambda, \widetilde{V}]. \quad (\text{C.14})$$

Inserting the gauge transformation  $\delta\widetilde{V}$ ,  $\Psi_\delta[\widetilde{V}]$  becomes

$$\begin{aligned}\Psi_\delta[\widetilde{V}] &= \pi_1\widehat{\mathbf{G}}(\delta\widetilde{V} \wedge e^{\wedge\eta\widetilde{V}}) \\ &= -\pi_1\widehat{\mathbf{G}}((\eta\xi\omega) \wedge e^{\wedge\eta\widetilde{V}}) - \pi_1\widehat{\mathbf{G}}\left((\pi_1\mathbf{L}(\xi\lambda \wedge e^{\wedge\eta\widetilde{V}})) \wedge e^{\wedge\eta\widetilde{V}}\right) \\ &= -\pi_1\mathbf{L}^\eta\left(\pi_1\widehat{\mathbf{G}}(\xi\omega \wedge e^{\wedge\eta\widetilde{V}}) \wedge e^{\wedge\pi_1\widehat{\mathbf{G}}(e^{\wedge\eta\widetilde{V}})}\right) - Q\pi_1\widehat{\mathbf{G}}(\xi\lambda \wedge e^{\wedge\eta\widetilde{V}}) - \Delta_T[\lambda, \eta\widetilde{V}] \\ &= D_\eta\Omega[\omega, \widetilde{V}] + Q\Lambda[\lambda, \widetilde{V}] - \Delta_T[\lambda, \eta\widetilde{V}].\end{aligned}\quad (\text{C.15})$$

The computation here is parallel to (2.65). Since  $\Delta_T$  is a trivial gauge transformation as in section 2.4,

$$\Delta_T[\lambda, \eta\widetilde{V}] = \pi_1\widehat{\mathbf{G}}(\xi\lambda \wedge \pi_1\mathbf{L}(e^{\wedge\eta\widetilde{V}}) \wedge e^{\wedge\eta\widetilde{V}}), \quad (\text{C.16})$$

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<sup>8</sup> More generally, the second term can be written by  $-\pi_1\boldsymbol{\xi}_Q(\lambda \wedge e^{\wedge\eta\widetilde{V}})$ . The representation above is only a choice of  $\boldsymbol{\xi}_d$ . However, the difference of the choices becomes  $\eta$ -exact term and it can be absorbed into  $-\eta\xi\omega$ .

we conclude that the gauge transformations of the trivial embedded theory (C.13) are written in the WZW-like form (C.14), where the gauge parameters  $\Omega$  and  $\Lambda$  are parameterized as

$$\Omega[\omega, \tilde{V}] = -\pi_1 \hat{\mathbf{G}}(\xi\omega \wedge e^{\wedge\eta\tilde{V}}), \quad (\text{C.17})$$

$$\Lambda[\lambda, \tilde{V}] = -\pi_1 \hat{\mathbf{G}}(\xi\lambda \wedge e^{\wedge\eta\tilde{V}}). \quad (\text{C.18})$$

In particular, one can find that the gauge transformations generated by  $\eta$  and  $Q$  do not mix: the gauge parameters  $\lambda$  and  $\omega$  of (C.13) are related to  $\Lambda$  and  $\Omega$  of (C.14) respectively, as seen in (C.17) and (C.18).

## D Open superstrings with stubs

In this section, we construct a new action for generic open NS string field theory, which we call alternative WZW-like action  $S_\eta$ . After defining the dual  $A_\infty$  products and explaining its WZW-like structure, we give two realizations of this type of WZW-like action using two different dynamical string fields  $\Phi$  and  $\Psi$  in the large and small Hilbert spaces, respectively.

### Dual $A_\infty$ -products and Derivation properties

Let  $\eta$  be the coderivation constructed from  $\eta$ , which is nilpotent  $\eta^2 = 0$ , and let  $\mathbf{a}$  be a nilpotent coderivation satisfying  $\mathbf{a}\eta = -(-)^{a\eta}\eta\mathbf{a}$  and  $\mathbf{a}^2 = 0$ . Then, we assume that  $\hat{\mathbf{G}}^{-1} : (\mathcal{H}, \mathbf{a}) \rightarrow (\mathcal{H}_S, \mathbf{D}_\mathbf{a})$  is an  $A_\infty$ -morphism, where  $\mathcal{H}$  is the large Hilbert space,  $\mathcal{H}_S$  is the small Hilbert space, and  $\mathbf{D}_\mathbf{a} \equiv \hat{\mathbf{G}}^{-1}\mathbf{a}\hat{\mathbf{G}}$ . Note that  $\mathbf{D}_\mathbf{a}$  is nilpotent:  $\mathbf{D}_\mathbf{a}^2 = (\hat{\mathbf{G}}^{-1}\mathbf{a}\hat{\mathbf{G}})(\hat{\mathbf{G}}^{-1}\mathbf{a}\hat{\mathbf{G}}) = \hat{\mathbf{G}}^{-1}\mathbf{a}^2\hat{\mathbf{G}} = 0$ . For example, one can use  $\mathbf{Q}$ ,  $\mathbf{Q} + \mathbf{m}_2|_2$ , and so on for  $\mathbf{a}$ , and various  $\hat{\mathbf{G}}$  appearing in [6–8] for  $\hat{\mathbf{G}}$ . Suppose that the coderivation  $\mathbf{D}_\mathbf{a}$  also commutes with  $\eta$ , which means

$$(\mathbf{D}_\mathbf{a})^2 = 0, \quad \llbracket \mathbf{D}_\mathbf{a}, \eta \rrbracket = 0. \quad (\text{D.1})$$

Then, we can introduce a dual  $A_\infty$ -products  $\mathbf{D}_\eta$  defined by

$$\mathbf{D}^\eta \equiv \hat{\mathbf{G}} \eta \hat{\mathbf{G}}^{-1}. \quad (\text{D.2})$$

Note that the pair of nilpotent maps  $(\mathbf{D}^\eta, \mathbf{a})$  have the same properties as  $(\mathbf{D}_\mathbf{a}, \eta)$ :

$$(\mathbf{D}^\eta)^2 = 0, \quad \llbracket \mathbf{D}^\eta, \mathbf{a} \rrbracket = 0. \quad (\text{D.3})$$

We can quickly find when the  $A_\infty$ -products  $\mathbf{D}_\mathbf{a}$  commutes with the coderivation  $\eta$  as (D.1), its dual  $A_\infty$ -product  $\mathbf{D}^\eta$  and coderivation  $\mathbf{a}$  also satisfies (D.3) as follows

$$\begin{aligned} \mathbf{a}\mathbf{D}^\eta &= (\hat{\mathbf{G}} \hat{\mathbf{G}}^{-1}) \mathbf{a} (\hat{\mathbf{G}} \eta \hat{\mathbf{G}}^{-1}) = \hat{\mathbf{G}} \mathbf{D}_\mathbf{a} \eta \hat{\mathbf{G}}^{-1} \\ &= (-)^{a\eta} \hat{\mathbf{G}} \eta \mathbf{D}_\mathbf{a} \hat{\mathbf{G}}^{-1} = (-)^{a\eta} \hat{\mathbf{G}} \eta \hat{\mathbf{G}}^{-1} \mathbf{a} \hat{\mathbf{G}} \hat{\mathbf{G}}^{-1} = (-)^{a\eta} \mathbf{D}^\eta \mathbf{a}. \end{aligned}$$

In this paper, as these coderivation  $\mathbf{a}$  and  $A_\infty$ -morphism  $\hat{\mathbf{G}}$ , we always use  $\mathbf{a} \equiv \mathbf{Q}$  and a gauge product  $\hat{\mathbf{G}}$  given by [7]. Therefore, the original  $A_\infty$  product is equal to the Neveu-Schwarz products of open strings with stubs given [7]

$$\mathbf{D}_\mathbf{a} = \mathbf{M} \equiv \hat{\mathbf{G}}^{-1} \mathbf{Q} \hat{\mathbf{G}}$$

and the dual  $A_\infty$  products is always given by

$$\mathbf{D}^\eta \equiv \widehat{\mathbf{G}} \boldsymbol{\eta} \widehat{\mathbf{G}}^{-1} = \boldsymbol{\eta} - \mathbf{m}_2 + \dots, \quad (\text{D.4})$$

and the symbol  $\mathbf{D}^\eta$  always denotes (D.4) in the rest. The dual  $A_\infty$  products  $\mathbf{D}^\eta$  consists of a linear map  $\boldsymbol{\eta}$  and a set of multilinear products  $\{D_n^\eta\}_{n=2}^\infty$ . Now the bilinear product  $D_2^\eta$  just equals to the star product  $m_2$  because we take  $\widehat{\mathbf{G}}$  of [7]. We write

$$[B_1, \dots, B_n]^\eta \equiv D_n^\eta(B_1 \otimes \dots \otimes B_n) \quad (\text{D.5})$$

for higher products of  $\mathbf{D}^\eta$ . Then, the Maurer-Cartan element of  $\mathbf{D}^\eta = \boldsymbol{\eta} - \mathbf{m}_2 + \dots$  is given by

$$\begin{aligned} \mathbf{D}^\eta \frac{1}{1-A} &= \frac{1}{1-A} \otimes \pi_1 \left( \mathbf{D}^\eta \frac{1}{1-A} \right) \otimes \frac{1}{1-A} \\ &= \frac{1}{1-A} \otimes \left( \boldsymbol{\eta} A - \mathbf{m}_2(A, A) + \dots \right) \otimes \frac{1}{1-A}, \end{aligned}$$

where  $A$  is a state of the large Hilbert space  $\mathcal{H}$  and  $\pi_1$  is a natural 1-state projection onto  $\mathcal{H}$ . Hence, the solution of the Maurer-Cartan equation  $\mathbf{D}^\eta(1-A)^{-1} = 0$  is given by a state  $\tilde{A}_\eta$  satisfying  $\boldsymbol{\eta} A_\eta - \mathbf{m}_2(A_\eta, A_\eta) + \dots = 0$ , or equivalently,

$$\boldsymbol{\eta} A_\eta - m_2(A_\eta, A_\eta) + \sum_{n=3}^{\infty} [\overbrace{A_\eta, \dots, A_\eta}^n]^\eta = 0.$$

#### Shift of the dual $A_\infty$ products $\mathbf{D}^\eta$

We introduce the  $A_\eta$ -shifted products  $[B_1, \dots, B_n]_{A_\eta}^\eta$  defined by

$$[B_1, \dots, B_n]_{A_\eta}^\eta \equiv \pi_1 \mathbf{D}^\eta \left( \frac{1}{1-A_\eta} \otimes B_1 \otimes \frac{1}{1-A_\eta} \otimes \dots \otimes \frac{1}{1-A_\eta} \otimes B_n \otimes \frac{1}{1-A_\eta} \right). \quad (\text{D.6})$$

Note that higher shifted products all vanish  $[B_1, \dots, B_{n>2}]_{A_\eta}^\eta = 0$  when higher products of  $\mathbf{D}^\eta = \boldsymbol{\eta} - m_2 + D_3^\eta + D_4^\eta \dots$  vanish:  $D_{n>2}^\eta = 0$ . In particular, we write  $D_\eta B$  for  $[B]_\eta^\eta$ :

$$\begin{aligned} D_\eta B &\equiv \pi_1 \mathbf{D}^\eta \left( \frac{1}{1-A_\eta} \otimes B \otimes \frac{1}{1-A_\eta} \right) \\ &= \boldsymbol{\eta} B - \llbracket A_\eta, B \rrbracket + \sum_{\text{cyclic } n=2}^{\infty} \sum [\overbrace{A_\eta, \dots, A_\eta}^n, B]^\eta. \end{aligned}$$

When  $\tilde{A}_\eta$  gives a solution on the Maurer-Cartan equation of  $\mathbf{D}_\eta$ , these  $A_\eta$ -shifted products also satisfy  $A_\infty$ -relations, which implies that the linear operator  $D_\eta$  becomes nilpotent. We find

$$\begin{aligned} (D_\eta)^2 B &= \pi_1 \mathbf{D}^\eta \left( \frac{1}{1-A_\eta} \otimes \pi_1 \mathbf{D}^\eta \left( \frac{1}{1-A_\eta} \otimes B \otimes \frac{1}{1-A_\eta} \right) \otimes \frac{1}{1-A_\eta} \right) \\ &= \pi_1 (\mathbf{D}^\eta)^2 \frac{1}{1-A_\eta} - \llbracket \pi_1 \mathbf{D}^\eta \left( \frac{1}{1-A_\eta} \right), B \rrbracket_{A_\eta}^\eta = 0, \end{aligned}$$

where the  $A_\eta$ -shifted commutator  $\llbracket A, B \rrbracket_{A_\eta}$  of  $A$  and  $B$  is defined by

$$\llbracket A, B \rrbracket_{A_\eta}^\eta \equiv [A, B]_{A_\eta}^\eta - (-)^{AB} [B, A]_{A_\eta}^\eta.$$



## WZW-like action $S_\eta[\varphi]$

We call a state  $A_\eta$  a pure-gauge-like field when  $A_\eta$  satisfies

$$\eta A_\eta - m_2(A_\eta, A_\eta) + \sum_{n=2}^{\infty} [\overbrace{A_\eta, \dots, A_\eta}^n]^\eta = 0 \quad (\text{D.7})$$

and has ghost-and-picture number  $(1| - 1)$ , odd Grassmann parity, and even degree. Let  $d$  is a derivation operator commuting with  $\eta$  and let  $(d_g|d_p)$  be its ghost-and-picture number. We call a state  $A_d$  a associated field when  $A_d$  satisfies

$$(-)^d d A_\eta = D_\eta A_d \quad (\text{D.8})$$

and has ghost-and-picture number  $(d_g - 1|d_p + 1)$ .

Let  $\varphi$  be a dynamical string field,  $t \in [0, 1]$  be a real parameter, and  $\varphi(t)$  be a path satisfying  $\varphi(0) = 0$  and  $\varphi(1) = \varphi$ . Once a pure-gauge-like field  $A_\eta = A_\eta[\varphi]$  and an associated field  $A_d = A_d[\varphi]$  are constructed as functionals of given dynamical string field  $\varphi$ , one can construct a gauge invariant action

$$S_\eta[\varphi] = - \int_0^1 dt \langle A_t[\varphi(t)], Q A_\eta[\varphi(t)] \rangle. \quad (\text{D.9})$$

We write  $A_t[\varphi]$  for the associated field  $A_d[\varphi]$  with the parameter differential  $d = \partial_t$ . One can check that the  $t$ -dependence is “topological”: the variation of the action is given by

$$\delta S_\eta[\varphi] = \langle A_\delta[\varphi], Q A_\eta[\varphi] \rangle.$$

Since  $Q A_\eta = -D_\eta A_Q$  and  $(D_\eta)^2 = Q^2 = 0$ , we find that  $\delta S = 0$  with the gauge transformations

$$A_\delta[\varphi] = D_\eta \Omega + Q \Lambda, \quad (\text{D.10})$$

where gauge parameters  $\Omega$  and  $\Lambda$  have ghost-and-picture number  $(-1|1)$  and  $(-1|0)$  respectively, and belong to the large Hilbert space. Since  $A_\delta[\varphi]$  is an invertible functional of  $\delta\varphi$ , the explicit form of the gauge transformation of the dynamical string field  $\delta\phi$  is obtained from  $A_\delta[\varphi]$ .

## Large-space parametrization: $\varphi = \Phi$

Let us consider an NS dynamical string field  $\Phi$  which is a Grassmann even, ghost number 0, and picture number 0 state of the large Hilbert space:  $\eta \Phi \neq 0$ . We show that one can construct WZW-like ingredients  $A_\eta[\Phi]$  and  $A_d[\Phi]$  as functionals of this dynamical string field  $\Phi$ . Using these WZW-like ingredients  $A_\eta[\Phi], A_d[\Phi]$  parametrized by  $\Phi$ , we obtain the WZW-like action  $S[\Phi]$  parametrized by  $\Phi$ :

$$S_\eta[\Phi] = - \int_0^1 dt \langle A_t[\Phi(t)], Q A_\eta[\Phi(t)] \rangle, \quad (\text{D.11})$$

where  $t$  is a real parameter  $t \in [0, 1]$  and  $\Phi(t)$  is the path satisfying  $\Phi(t = 0) = 0$  and  $\Phi(t = 1) = \Phi$ . We wrote  $A_t[\Psi]$  for the associated field  $A_d[\Psi]$  with  $d = \partial_t$ .

### Pure-gauge-like field $A_\eta[\Phi]$

Let us consider a functional  $A_\eta[\tau; \Phi]$  defined by the differential equation

$$\begin{aligned} \frac{\partial}{\partial \tau} A_\eta[\tau; \Phi] &= \eta \Phi + \sum_{k=1}^{\infty} \sum_{\text{cyclic}} [\overbrace{A_\eta[\tau; \Phi], \dots, A_\eta[\tau; \Phi]}^k, \Phi]^\eta \\ &\equiv D_\eta(\tau) \Phi. \end{aligned} \quad (\text{D.12})$$

with the initial condition  $A_\eta[\tau = 0; \Phi] = 0$ , where  $\tau$  is a real parameter. A few terms of  $A_\eta[\tau; \Phi]$  is given by

$$A_\eta[\tau; \Phi] = \tau \eta \Phi + \frac{\tau^2}{2} \llbracket \eta \Phi, \Phi \rrbracket + \frac{\tau^3}{3} \left( \sum_{\text{cyclic}} [\eta \Phi, \eta \Phi, \Phi]^\eta + \llbracket \llbracket \eta \Phi, \Phi \rrbracket^\eta, \Phi \rrbracket^\eta \right) + \dots$$

We write  $A_\eta[\Phi]$  for the  $\tau = 1$  value of the solution  $A_\eta[\tau; \Phi]$ :

$$A_\eta[\Phi] \equiv A_\eta[\tau = 1; \Phi].$$

We quickly find that this  $A_\eta[\Phi]$ , a functional on the dynamical string field  $\Phi$ , satisfies (refp-eq. A) and gives a pure-gauge-like field. For brevity, we introduce

$$F(\tau) \equiv \eta A_\eta[\tau; \Phi] - m_2(A_\eta[\tau; \Phi], A_\eta[\tau; \Phi]) + \sum_{n=2}^{\infty} [\overbrace{A_\eta[\tau; \Phi], \dots, A_\eta[\tau; \Phi]}^n]^\eta.$$

The statement “ $A_\eta[\Phi]$  is a pure-gauge-like field” is equivalent to the equation  $F(1) = 0$ . By definition,  $F(\tau)$  satisfies  $F(\tau = 0) = 0$  and the following relation holds:

$$\begin{aligned} \frac{\partial}{\partial \tau} F(\tau) &= \eta \frac{\partial}{\partial \tau} A_\eta[\tau; \Phi] + \sum_{n=1}^{\infty} \sum_{\text{cyclic}} [\overbrace{A_\eta[\tau; \Phi], \dots, A_\eta[\tau; \Phi]}^n, \frac{\partial}{\partial \tau} A_\eta[\tau; \Phi]]^\eta \\ &= D_\eta(\tau) D_\eta(\tau) \Phi = -\llbracket F(\tau), \Phi \rrbracket_{A_\eta[\tau; \Phi]}^\eta. \end{aligned} \quad (\text{D.13})$$

The solution of the differential equation (D.13) with the initial condition  $F(0) = 0$  is given by  $F(\tau) = 0$  for any  $\tau$ . Hence,  $F(1) = 0$  holds and  $A_\eta[\Phi]$  indeed gives a pure-gauge-like field.

### Associated field $A_d[\Phi]$

Then, we consider a functional  $A_d[\tau; \Phi]$  defined by the differential equation

$$\frac{\partial}{\partial \tau} A_d[\tau; \Phi] = d\Phi + \llbracket \Phi, A_d[\tau; \Phi] \rrbracket_{A_\eta[\tau; \Phi]}^\eta \quad (\text{D.14})$$

with the initial condition  $A_d[\tau = 0; \Phi] = 0$ . A few terms of  $A_d[\tau; \Phi]$  is given by

$$A_d[\tau; \Phi] = \tau d\Phi + \frac{\tau^2}{2} \llbracket d\Phi, \Phi \rrbracket + \frac{\tau^3}{3} \left( \sum_{\text{cyclic}} [\eta \Phi, d\Phi, \Phi]^\eta + \frac{1}{2} \llbracket \llbracket d\Phi, \Phi \rrbracket^\eta, \Phi \rrbracket^\eta \right) + \dots$$

The  $\tau = 1$  value of the functional  $A_d[\tau; \Phi]$  of the dynamical string field  $\Phi$ , which we write

$$A_d[\Phi] \equiv A_d[\tau = 1; \Phi],$$

satisfies (D.8) and gives an associated field. To prove this fact, we introduce a function of  $\tau$

$$\mathcal{I}(\tau) \equiv D_\eta(\tau)A_d[\tau; \Phi] - (-)^d A_\eta[\tau; \Phi],$$

whose zeros would provide the WZW-like relation that associated field must satisfy. By definition,  $\mathcal{I}(\tau)$  satisfies  $\mathcal{I}(\tau = 0) = 0$  and

$$\begin{aligned} \frac{\partial}{\partial \tau} \mathcal{I}(\tau) &= D_\eta(\tau) (\partial_\tau A_d[\tau; \Phi]) + [\partial_\tau A_\eta[\tau; \Phi], A_d[\tau; \Phi]]_{A_\eta[\tau; \Phi]}^\eta - (-)^d d(\partial_\tau A_\eta[\tau; \Phi]) \\ &= [\Phi, \mathcal{I}(\tau)]_{A_\eta[\tau; \Phi]}^\eta + D_\eta \left( \frac{\partial}{\partial \tau} A_d[\tau; \Phi] - d\Phi - \llbracket \Phi, A_d[\tau; \Phi] \rrbracket_{A_\eta[\tau; \Phi]}^\eta \right). \end{aligned}$$

Hence, when  $A_d[\tau; \Phi]$  is defined by (D.14), the function  $\mathcal{I}(\tau)$  gives the solution of the differential equation

$$\frac{\partial}{\partial \tau} \mathcal{I}(\tau) = \llbracket \Phi, \mathcal{I}(\tau) \rrbracket_{A_\eta[\tau; \Phi]}^\eta$$

with the initial condition  $\mathcal{I}(0) = 0$ , namely,  $\mathcal{I}(\tau) = 0$  for any  $\tau$ : we obtain  $\mathcal{I}(1) = 0$ .

### Small-space parametrization: $\varphi = \Psi$

Let us consider an NS dynamical string field  $\Psi$  which is a Grassmann odd, ghost number 1, and picture number  $-1$  state of the small Hilbert space:  $\eta \Psi = 0$ . We show that one can construct WZW-like ingredients  $A_\eta[\Psi]$  and  $A_d[\Psi]$  as functionals of this dynamical string field  $\Psi$ . Once we obtain these WZW-like ingredients  $A_\eta[\Psi], A_d[\Psi]$  parametrized by  $\Psi$ , one can construct the WZW-like action  $S[\Psi]$  parametrized by  $\Psi$ :

$$S_\eta[\Psi] = - \int_0^1 ds \langle A_s[\Psi(s)], Q A_\eta[\Psi(s)] \rangle, \quad (\text{D.15})$$

where  $s$  is a real parameter  $s \in [0, 1]$  and  $\Psi(s)$  is the path satisfying  $\Psi(s = 0) = 0$  and  $\Psi(s = 1) = \Psi$ . We wrote  $A_s[\Psi]$  for the associated field  $A_d[\Psi]$  with  $d = \partial_s$ .

#### Pure-gauge-like field $A_\eta[\Psi]$

When the  $\boldsymbol{\eta}$ -complex  $(\mathcal{H}, \boldsymbol{\eta})$  is exact, there exist  $\boldsymbol{\xi}$  such that  $\llbracket \boldsymbol{\eta}, \boldsymbol{\xi} \rrbracket = \mathbf{1}$  and  $\mathcal{H}$ , the large Hilbert space, is decomposed into the direct sum of  $\eta$ -exact and  $\xi$ -exact  $\mathcal{H} = P_\eta \mathcal{H} \oplus P_\xi \mathcal{H}$ , where  $P_\eta$  and  $P_\xi$  are projector onto  $\eta$ -exact and  $\xi$ -exact states respectively.<sup>9</sup> Note that since the small Hilbert space  $\mathcal{H}_S$  is defined by  $\mathcal{H}_S \equiv P_\eta \mathcal{H}$  and satisfies  $\mathcal{H}_S \subset P_\eta \mathcal{H}_S$ , all the states  $\Psi$  belonging to  $\mathcal{H}_S$  satisfy  $P_\eta \Psi = \Psi$  and  $P_\xi \Psi = 0$ , or simply,

$$\boldsymbol{\eta} \Psi = 0.$$

Using this fact, we can construct a desired pure-gauge-like fields  $A_\eta[\Psi]$ , namely, a solution of the Maurer-Cartan equation of the dual  $A_\infty$  products  $\boldsymbol{D}^\eta = \boldsymbol{\eta} - \boldsymbol{m}_2 + \dots$ .

A pure-gauge-like field  $A_\eta[\Psi]$  is given by

$$A_\eta[\Psi] \equiv \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \Psi}$$

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<sup>9</sup>These satisfy  $P_\eta^2 = P_\eta$ ,  $P_\xi^2 = P_\xi$ ,  $P_\eta P_\xi = P_\xi P_\eta = 0$ , and  $P_\eta + P_\xi = 1$  on  $\mathcal{H}$ .

because it becomes a trivial solution of the Maurer-Cartan equation as follows

$$\begin{aligned} D^\eta \frac{1}{1 - A_\eta[\Psi]} &= D^\eta \frac{1}{1 - \pi_1 \widehat{\mathbf{G}} \frac{1}{1 - \Psi}} = D^\eta \widehat{\mathbf{G}} \frac{1}{1 - \Psi} = \widehat{\mathbf{G}} \eta \frac{1}{1 - \Psi} \\ &= \widehat{\mathbf{G}} \left( \frac{1}{1 - \Psi} \otimes \eta \Psi \otimes \frac{1}{1 - \Psi} \right) = 0. \end{aligned}$$

Recall that  $\pi_1 D^\eta (1 - A_\eta[\Psi])^{-1} = 0$  is equal to

$$\eta A_\eta[\Psi] - m_2(A_\eta[\Psi], A_\eta[\Psi]) + \sum_{n=2}^{\infty} [\overbrace{A_\eta[\Psi], \dots, A_\eta[\Psi]}^n]^\eta = 0.$$

### Associated fields $A_d[\Psi]$

Let  $\mathbf{d}$  be a coderivation constructed from a derivation  $d$  of the dual  $A_\infty$ -products  $D^\eta$ , which implies that the  $d$ -derivation property  $[[\mathbf{d}, D^\eta]] = 0$  holds. Then, we obtain  $[[D_d, \eta]] = 0$  with  $D_d \equiv \widehat{\mathbf{G}}^{-1} \mathbf{d} \widehat{\mathbf{G}}$ , which means that  $D_d$  is “ $\eta$ -exact” and there exists a coderivation  $\xi_d$  such that

$$D_d = \widehat{\mathbf{G}}^{-1} \mathbf{d} \widehat{\mathbf{G}} = (-)^d [[\eta, \xi_d]].$$

Using this coderivation  $\xi_d$ , we can construct an associated field  $A_d[\Psi]$ . Note that the response of  $\mathbf{d}$  acting on the group-like element of  $A_\eta[\Psi] = \widehat{\mathbf{G}}(1 - \Psi)^{-1}$  is given by

$$\begin{aligned} (-)^d \mathbf{d} \frac{1}{1 - A_\eta[\Psi]} &= (-)^d \widehat{\mathbf{G}} \widehat{\mathbf{G}}^{-1} \mathbf{d} \widehat{\mathbf{G}} \frac{1}{1 - \Psi} = \widehat{\mathbf{G}} \eta \xi_d \frac{1}{1 - \Psi} = D^\eta \widehat{\mathbf{G}} \left( \xi_d \frac{1}{1 - \Psi} \right) \\ &= D^\eta \left( \frac{1}{1 - A_\eta[\Psi]} \otimes \pi_1 \widehat{\mathbf{G}} \left( \xi_d \frac{1}{1 - \Psi} \right) \otimes \frac{1}{1 - A_\eta[\Psi]} \right). \end{aligned}$$

An associated field of  $d$  is given by

$$A_d[\Psi] \equiv \pi_1 \widehat{\mathbf{G}} \left( \xi_d \frac{1}{1 - \Psi} \right)$$

because one can directly check

$$\begin{aligned} (-)^d \mathbf{d} \frac{1}{1 - A_\eta[\Psi]} &= D^\eta \left( \frac{1}{1 - A_\eta[\Psi]} \otimes \pi_1 \widehat{\mathbf{G}} \left( \xi_d \frac{1}{1 - \Psi} \right) \otimes \frac{1}{1 - A_\eta[\Psi]} \right) \\ &= D^\eta \left( \frac{1}{1 - A_\eta[\Psi]} \otimes A_d[\Psi] \otimes \frac{1}{1 - A_\eta[\Psi]} \right). \end{aligned}$$

Picking up the relation on  $\mathcal{H}$ , or equivalently acting  $\pi_1$  on this relation on  $T(\mathcal{H})$ , we obtain

$$(-)^d d A_\eta[\Psi] = \eta A_d[\Psi] - [[A_\eta[\Psi], A_d[\Psi]]]_{A_\eta[\Psi]}^\eta,$$

which is the simplest case of  $(-)^d d A_\eta[\Psi] = \pi_1 D^\eta$ -exact term.

## E NS-NS sector

In this section, we clarify a WZW-like structure naturally arising from  $L_\infty$  action and give alternative WZW-like form  $S_{\eta\tilde{\eta}}$  of the action for NS-NS string field theory.

## Dual $L_\infty$ products

Using path-ordered exponential map  $\widehat{\mathbf{G}}$ , the small-space closed NS-NS superstring products is given by  $\mathbf{L} = \widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}$ , which satisfies  $[\![\boldsymbol{\eta}, \mathbf{L}]\!] = 0$  and  $[\![\widetilde{\boldsymbol{\eta}}, \mathbf{L}]\!] = 0$ . We start with its dual  $L_\infty$  products

$$\mathbf{L}^\eta = \widehat{\mathbf{G}} \boldsymbol{\eta} \widehat{\mathbf{G}}^{-1}, \quad (\text{E.1})$$

$$\mathbf{L}^{\widetilde{\eta}} = \widehat{\mathbf{G}} \widetilde{\boldsymbol{\eta}} \widehat{\mathbf{G}}^{-1}, \quad (\text{E.2})$$

satisfying  $L_\infty$ -relations and  $Q$ -derivation properties

$$\mathbf{Q} \mathbf{L}^\alpha = \widehat{\mathbf{G}} (\widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}) \alpha \widehat{\mathbf{G}}^{-1} = -\widehat{\mathbf{G}} \alpha (\widehat{\mathbf{G}}^{-1} \mathbf{Q} \widehat{\mathbf{G}}) \widehat{\mathbf{G}}^{-1} = -\mathbf{L}^\alpha \mathbf{Q}$$

for  $\alpha = \boldsymbol{\eta}, \widetilde{\boldsymbol{\eta}}$ , which give extensions of  $\eta$ -constraints. We write

$$[A_1, \dots, A_n]^\alpha := \pi_1 \widehat{\mathbf{G}} \alpha \widehat{\mathbf{G}}^{-1} (A_1 \wedge \dots \wedge A_n), \quad (\alpha = \boldsymbol{\eta}, \widetilde{\boldsymbol{\eta}}). \quad (\text{E.3})$$

By definition, dual  $L_\infty$  products for  $\alpha = \boldsymbol{\eta}, \widetilde{\boldsymbol{\eta}}$  satisfy  $L_\infty$ -relations, have  $Q$ -derivation properties, and commute with each other:

$$\sum_\sigma \sum_{k=1}^n (-)^{|\sigma|} [[A_{i_{\sigma(1)}}, \dots, A_{i_{\sigma(k)}}]^\alpha, A_{i_{\sigma(k+1)}}, \dots, A_{i_{\sigma(n)}}]^\alpha = 0, \quad (\text{E.4})$$

$$Q[B_1, \dots, B_n]^\alpha + \sum_{i=1}^{n-1} (-)^{B_1 + \dots + B_{k-1}} [B_1, \dots, QB_k, \dots, B_n]^\alpha = 0, \quad (\text{E.5})$$

$$\sum_{\alpha_1, \alpha_2 = \boldsymbol{\eta}, \widetilde{\boldsymbol{\eta}}} \sum_\sigma \sum_{k=1}^n (-)^{|\sigma|} [[A_{i_{\sigma(1)}}, \dots, A_{i_{\sigma(k)}}]^{\alpha_1}, A_{i_{\sigma(k+1)}}, \dots, A_{i_{\sigma(n)}}]^{\alpha_2} = 0. \quad (\text{E.6})$$

## Wess-Zumino-Witten-like relations

A pure-gauge-like field  $\Psi_{\boldsymbol{\eta}\widetilde{\boldsymbol{\eta}}}$  is a Grassmann even, ghost number 2, left-moving picture number  $-1$ , and right-moving picture number  $-1$  state in the left-and-right large Hilbert space. This  $\Psi_{\boldsymbol{\eta}\widetilde{\boldsymbol{\eta}}}$  satisfies the Maurer-Cartan equations for the both dual products (E.1) and (E.2):

$$\alpha \Psi_{\boldsymbol{\eta}\widetilde{\boldsymbol{\eta}}} + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} [\overbrace{\Psi_{\boldsymbol{\eta}\widetilde{\boldsymbol{\eta}}}, \dots, \Psi_{\boldsymbol{\eta}\widetilde{\boldsymbol{\eta}}}}^{n+1}]^\alpha = 0, \quad (\alpha = \boldsymbol{\eta}, \widetilde{\boldsymbol{\eta}}). \quad (\text{E.7})$$

The starting point of our WZW-like theory is now to find this pure-gauge-like field  $\Psi_{\boldsymbol{\eta}\widetilde{\boldsymbol{\eta}}}$  and its associated fields  $\Psi_{\eta d}$ ,  $\Psi_{d\widetilde{\eta}}$ , and  $\Psi_d^*$  which satisfy

$$(-)^d d \Psi_{\boldsymbol{\eta}\widetilde{\boldsymbol{\eta}}} = D_\eta \Psi_{d\widetilde{\eta}} = D_{\widetilde{\eta}} \Psi_{\eta d}, \quad (\text{E.8})$$

$$D_\eta \Psi_d^* = \Psi_{\eta d}, \quad -D_{\widetilde{\eta}} \Psi_d^* = \Psi_{d\widetilde{\eta}} \quad (\text{E.9})$$

where the linear operator  $D_\alpha$  for  $\alpha = \boldsymbol{\eta}, \widetilde{\boldsymbol{\eta}}$  is given by

$$D_\alpha B \equiv \alpha B + \sum_{n=1}^{\infty} \frac{1}{n!} [\overbrace{\Psi_{\boldsymbol{\eta}\widetilde{\boldsymbol{\eta}}}, \dots, \Psi_{\boldsymbol{\eta}\widetilde{\boldsymbol{\eta}}}}^n, B]^\alpha, \quad (\alpha = \boldsymbol{\eta}, \widetilde{\boldsymbol{\eta}}), \quad (\text{E.10})$$

and  $d$  is a derivation operator for the alternative products

$$(-)^d d [B_1, \dots, B_n]^\alpha = \sum_{k=1}^n (-)^{d(B_1 + \dots + B_{k-1})} [B_1, \dots, dB_k, \dots, B_n]^\alpha, \quad (\alpha = \eta, \tilde{\eta}). \quad (\text{E.11})$$

Note that  $D_\eta D_{\tilde{\eta}} B = -D_{\tilde{\eta}} D_\eta B$  holds because  $\Psi_{\eta\tilde{\eta}}$  satisfies (E.7) for  $\alpha = \eta, \tilde{\eta}$ .

Let  $d$  has ghost number  $d_g$ , left-moving picture number  $d_p$ , and right-moving picture  $d_{\tilde{p}}$ . Then, these associated fields  $\Psi_{\eta d}$ ,  $\Psi_{d\tilde{\eta}}$ , and  $\Psi_d^*$  have ghost-and-picture number  $(1-d_g|d_p-1, d_{\tilde{p}})$ ,  $(1-d_g|d_p, d_{\tilde{p}}-1)$ , and  $(d_g|d_p, d_{\tilde{p}})$ , respectively, where  $(d_g|d_p, d_{\tilde{p}})$  implies that the ghost number is  $d_g$ , the left-moving picture number is  $d_p$ , and the right-moving picture is  $d_{\tilde{p}}$ . We call these two relations (E.7), (E.8), and (E.9) as *Wess-Zumino-Witten-like relations*.

### Alternative WZW-like action $S_{\eta\tilde{\eta}}[\varphi]$

Let  $\varphi$  be a dynamical NS-NS string field and  $\varphi(t)$  be a path satisfying  $\varphi(t=0) = 0$  and  $\varphi(t=1) = \varphi$ , where  $t \in [0, 1]$  is a real parameter. Once we construct WZW-like ingredients  $\Psi_{\eta\tilde{\eta}}[\varphi]$  and  $\Psi_d^*[\varphi]$  as functionals of given dynamical string field  $\varphi$ , we can obtain a new gauge invariant action

$$S_{\eta\tilde{\eta}}[\varphi] = \int_0^1 dt \langle \Psi_t^*[\varphi(t)], Q \Psi_{\eta\tilde{\eta}}[\varphi(t)] \rangle, \quad (\text{E.12})$$

whose gauge transformations are given by

$$\Psi_\delta^*[\varphi] = D_\eta \Omega + D_{\tilde{\eta}} \tilde{\Omega} + Q \Lambda. \quad (\text{E.13})$$

The equation of motion is given by  $t$ -independent form

$$Q \Psi_{\eta\tilde{\eta}}[\varphi] = -D_\eta \Psi_{Q\tilde{\eta}}[\varphi] = -D_{\tilde{\eta}} \Psi_{\eta Q}[\varphi] = D_\eta D_{\tilde{\eta}} \Psi_Q^* = 0. \quad (\text{E.14})$$

One can quickly find these facts by using WZW-like relations (E.7), (E.8), and (E.9) only.

### Small-space parametrization: $\varphi = \Phi$

Let  $\Phi$  be a NS-NS dynamical string field belonging to the small Hilbert space, which is a Grassmann even, ghost number 2, left-moving picture number  $-1$ , and right-moving picture number  $-1$  state. We show that the pure-gauge-like field  $\Psi_{\eta\tilde{\eta}}$  is given by

$$\Psi_{\eta\tilde{\eta}}(t) = \pi_1 \hat{\mathbf{G}}(e^{\wedge \Phi(t)}), \quad (\text{E.15})$$

and the linear operator  $D_\alpha$  for  $\alpha = \eta, \tilde{\eta}$  becomes

$$D_\alpha = \pi_1 \mathbf{L}^\alpha (\mathbb{I} \wedge e^{\wedge \pi_1 \hat{\mathbf{G}}(e^{\wedge \Phi})}), \quad (\alpha = \eta, \tilde{\eta}). \quad (\text{E.16})$$

Then, the left associated field  $\Psi_{d\tilde{\eta}}$  and the right associated field  $\Psi_{\eta d}$  are given by

$$\Psi_{d\tilde{\eta}}(t) = \pi_1 \hat{\mathbf{G}}(\xi_d e^{\wedge \Phi(t)}), \quad (\text{E.17})$$

$$\Psi_{\eta d}(t) = \pi_1 \hat{\mathbf{G}}(\tilde{\xi}_d e^{\wedge \Phi(t)}), \quad (\text{E.18})$$

respectively. The associated field  $\Psi_d^*$ , which we call the conjugate field, is given by

$$\begin{aligned} \Psi_d^* &= \pi_1 \hat{\mathbf{G}}(\xi \tilde{\xi} d\Phi(t) \wedge e^{\wedge \Phi(t)}) \\ &= \pi_1 \hat{\mathbf{G}}(\xi_{\tilde{\xi}_d} e^{\wedge \Phi(t)}) = -\pi_1 \hat{\mathbf{G}}(\tilde{\xi}_{\xi_d} e^{\wedge \Phi(t)}). \end{aligned} \quad (\text{E.19})$$

## Proofs of properties

Since the group-like element given by (E.15) satisfies

$$\mathbf{L}^\eta(e^{\pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)})}) = (\widehat{\mathbf{G}} \boldsymbol{\eta} \widehat{\mathbf{G}}^{-1}) \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}) = \widehat{\mathbf{G}} \boldsymbol{\eta}(e^{\wedge \Phi(t)}) = 0, \quad (\text{E.20})$$

we obtain the desired equation  $\pi_1 \mathbf{L}^\eta(e^{\wedge \Psi_\eta(t)}) = 0$ , or equivalently,

$$\eta \Psi_{\eta\tilde{\eta}}(t) + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} [\overbrace{\Psi_{\eta\tilde{\eta}}(t), \dots, \Psi_{\eta\tilde{\eta}}(t)}^{n+1}]^\eta = 0. \quad (\text{E.21})$$

Similarly, provided that  $d\widehat{\mathbf{G}} = \widehat{\mathbf{G}} d$  and  $\llbracket X, \eta \rrbracket = 0$ , after the following computation

$$\begin{aligned} (-)^X X \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}) &= (-)^d \widehat{\mathbf{G}}(d\Phi(t) \wedge e^{\wedge \Phi(t)}) \\ &= \widehat{\mathbf{G}} \boldsymbol{\eta}(\xi_d \Phi(t) \wedge e^{\wedge \Phi(t)}) \\ &= \mathbf{L}^\eta \widehat{\mathbf{G}}(\xi_d \Phi(t) \wedge e^{\wedge \Phi(t)}) \\ &= \mathbf{L}^\eta \left( \pi_1 \widehat{\mathbf{G}}(\xi_d \Phi(t) \wedge e^{\wedge \Phi(t)}) \wedge e^{\wedge \pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)})} \right), \end{aligned} \quad (\text{E.22})$$

we find that the second Wess-Zumino-Witten-like relation holds:

$$(-)^d d\Psi_{\eta\tilde{\eta}}(t) = D_\eta(t) \Psi_{d\tilde{\eta}}(t). \quad (\text{E.23})$$

Note that  $\llbracket Q, \eta \rrbracket = 0$  but  $\mathbf{Q} \widehat{\mathbf{G}} = \widehat{\mathbf{G}} \mathbf{L}$ . Utilizing the coderivation  $\boldsymbol{\xi}_Q$  such that  $\mathbf{L} = -\llbracket \boldsymbol{\eta}, \boldsymbol{\xi}_Q \rrbracket$ , we can check that the field

$$\Psi_{Q\tilde{\eta}}(t) = \pi_1 \widehat{\mathbf{G}}(\boldsymbol{\xi}_Q e^{\wedge \Phi(t)}) \quad (\text{E.24})$$

satisfies the second Wess-Zumino-Witten-like relation  $-Q \Psi_{\eta\tilde{\eta}} = D_\eta \Psi_{Q\tilde{\eta}}$  as follows

$$\begin{aligned} -\mathbf{Q} \widehat{\mathbf{G}}(e^{\wedge \Phi(t)}) &= -\widehat{\mathbf{G}} \mathbf{L}(e^{\wedge \Phi(t)}) \\ &= \widehat{\mathbf{G}} \boldsymbol{\eta} \boldsymbol{\xi}_Q(e^{\wedge \Phi(t)}) \\ &= \mathbf{L}^\eta \widehat{\mathbf{G}} \boldsymbol{\xi}_Q(e^{\wedge \Phi(t)}) \\ &= \mathbf{L}^\eta \left( \pi_1 \widehat{\mathbf{G}} \boldsymbol{\xi}_Q(e^{\wedge \Phi(t)}) \wedge e^{\wedge \pi_1 \widehat{\mathbf{G}}(e^{\wedge \Phi(t)})} \right). \end{aligned} \quad (\text{E.25})$$

As the almost same way, one can check  $\pi_1 \mathbf{L}^\eta e^{\wedge \Psi_{\eta\tilde{\eta}}} = 0$  and  $(-)^d d\Psi_{\eta\tilde{\eta}} = D_{\tilde{\eta}} \Psi_{\eta d}$ . Then, we find

$$\llbracket D_\eta, D_{\tilde{\eta}} \rrbracket B = -\pi_1 \mathbf{L}^\eta(\pi_1 \mathbf{L}^\eta(e^{\Psi_{\eta\tilde{\eta}}}) \wedge B \wedge e^{\Psi_{\eta\tilde{\eta}}}) - \pi_1 \mathbf{L}^\eta(\pi_1 \mathbf{L}^\eta(e^{\Psi_{\eta\tilde{\eta}}}) \wedge B \wedge e^{\Psi_{\eta\tilde{\eta}}}) \quad (\text{E.26})$$

and thus, (E.19) gives an appropriate conjugate field.

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